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AN ENQUIRY INTO THE PRINCIPAL PHÆNOMENA OF Sounds and Muſical Strings.

BY

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Advertiſement.

IN the following Diſcourſe I have endeavoured to vindicate the 47th prop. B. 2, of Newton's Principia from the objections which have been made againſt it; as it appears to me to be the only true principle on which the phænomena of the pulſes of air can be explained. I have alſo endeavoured to correct ſuch errors as, I apprehend, have been admitted into the theory of ſound; and have enquired with ſome minuteneſs into the nature of the vibrations of elaſtic fibres, a part of natural philoſophy which ſeems to be more fertile in conſequences than has hitherto been ſuppoſed, ſince the ingenious writings of Hartley and Prieſtley on that ſubject. The phænomena of muſical ſtrings are alſo accounted for by a theory which is at leaſt plauſible: and though it is not propoſed as a rigid demonſtration, yet the great variety of experiments which conſpire to confirm its truth, will probably be looked on as ſetting it far above conjecture.

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PART I.

ON SOUNDS.

THE phænomena of ſound are ſo intereſting and amuſing, and, at the ſame time, ſo ſurpriſing, that we cannot wonder they have been found amongſt the principal objects that have engaged the attention of philoſophers. The ancients, however, do not appear to have made any great progreſs in this branch of ſcience, conſidered as a ſubject of philoſophical enquiry. They cultivated theſe phænomena principally with a view to the improvement of muſic; and, except a few vague and inconcluſive theories, we ſcarcely find any thing conſiderable done till we come down to the age of the incomparable Newton. His ſagacity firſt explained the laws according to which the aerial pulſes are propagated, and opened the way to the proſecution of this intricate ſubject. But, notwithſtanding the attention which has been given to theſe diſcoveries, we find that they have received no acceſſion of any conſequence ſince his time.

The great truth which he diſcovered, and which is the foundation of almoſt all our reaſoning on ſound was, that "the pulſes of the air are propagated in all directions from the ſounding body," and "that during their progreſs and regreſs, they are twice accelerated and twice retarded, according to the law of a pendulum vibrating in a cycloid:" a truth not leſs ſubtle and aſtoniſhing than his diſcovery, that the refracted ſpectrum of the ſun's image is divided by the ſeveral confines of its ſeven colours, in the ſame manner as a muſical chord which produces the ſucceſſive notes of an octave.

I do not propoſe to lay down in the following pages a regular ſyſtem of the doctrine of ſound, as that has been done already by a great variety of authors: I ſhall only endeavour to correct ſuch errors as I conceive have been admitted into the preſent ſyſtem, and to explain and ſolve ſuch phæno-mena as do not appear to me to have been ſatisfactorily accounted for.

SECT. I.

OF THE PROPAGATION OF SOUND.

1. AIR is univerſally admitted to be the medium of ſound; the experiment by which it is proved is as follows: a bell being included within the receiver of an air-pump, according as the air is in part exhauſted, the ſound of the bell proportionally diminiſhes; and, when the proceſs has been ſo far continued as that a perfect evacuation has been nearly effected, the ſound at the ſame time ſeems entirely to ceaſe. It may be objected to the concluſiveneſs of this experiment, that when the air is exhauſted from within, the preſſure from the incumbent air on the exterior ſurface will prevent the ſound's being tranſmitted, in the ſame manner as a body is prevented from ſounding by ſuſtain-

ing a preſſure on any *part* of it. But this objection may be thus obviated: let the air in the receiver be condenſed inſtead of being exhauſted, and the ſound, ſo far from being extinguiſhed, will be encreaſed; though in this caſe there is as great an inequality of preſſure on the two ſurfaces as in the former.

2. In treating of the propagation of ſound through this medium, the generality of writers content themſelves with ſaying, that "the parts of the ſounding body, by going forward, compreſs the contiguous parts of the medium, which again compreſs the next, and ſo on; and that by returning, they ſuffer the compreſſed parts to expand; and, as theſe compreſſed parts will endeavour to expand themſelves as well towards the adjacent parts of the medium which are at reſt, as towards the rarer intervals of the pulſes, and the quie-

ſcent parts will alſo expand themſelves into theſe intervals, the motion will be propagated in all directions from the ſounding body*." So far their reaſoning may be admitted: that is, to ſatisfy us that the pulſes are propagated by ſucceſſive goings forward and returnings backward, and ſucceſſive condenſations and rarefactions of the medium. But when they proceed ſtill farther *, and pretend to draw from their arguing this immediate inference, that "the ſucceſſive goings and returnings of the parts of the medium obſerve exactly the ſame law with that of the ſounding body," we muſt heſitate with regard to the juſtneſs of the concluſion: for it does not appear impoſſible, but

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* Monſ. Hube, (*vide Acta Eruditorum, an.* 1762,) has endeavoured to refute this concluſion. His argument is to the following effect: "If a vertical tube, whoſe lower orifice is cloſed, be filled with water, the ſides will ſuſtain a lateral preſſure; but if the lower orifice be opened, ſo as that the water may freely flow out, the lateral preſſure will ceaſe. In the ſame manner, if air be included in a veſſel and compreſſed, its preſſure will extend to all the particles in the veſſel. But it does not thence follow, that the like ſhould happen in the propagation of ſound; for he that would reaſon ſo, muſt maintain, that the ſides of a vertical tube ſuſtain a preſſure from the water which freely paſſes thro' it." But this example, which he has alledged, is not to the purpoſe. If, indeed, the particles of air included in a tube were all moved forward with the ſame velocity, as he ſuppoſes of the particles of water, it is evident there would be no condenſation, and conſequently no increaſe of the lateral expanſion, which is all that his example tends to ſhew. But, if the preceding particles of water were retarded in their progreſs, ſo as that the ſubſequent particles ſhould preſs againſt them, it is evident that, in ſuch a caſe, there would be a lateral preſſure exerted on

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the ſides of the tube, proportionally greater as the velocity of the ſubſequent exceeded that of the preceding particles. Now this is the caſe, and this caſe only, which bears any analogy to the propagation of ſound; for in the aerial pulſes, the preceding particles, during the firſt half of their progreſs, move with a leſs velocity, than thoſe which immediately follow; a condenſation therefore is produced, and conſequently a lateral expanſion.

* See Helſham, lect. 17, Pemberton, Martin, Rutherford, &c.

that the elaſtic force of the medium may be of ſuch a nature, as to diſturb and vary that motion which has been impreſſed on it by the ſonorous body.

Mr. Rowning, indeed, has conſidered the conſtitution of the air: but the following defects appear in his reaſoning; he ſays, that "the velocity with which each particle moves is proportional to the force which repels it, and that this force is proportional to the diſtance of the particle from its proper place." Now in both theſe aſſertions he ſeems to me to have fallen into an error; for if the diameter of a circle repreſent the ſpace through which a particle vibrates, and its perimeter the time of the vibration, the force which actuates the particle will be always proportional to the coſine of the arch which repreſents the time of the motion; not to its diſtance from its proper place, which is the verſed ſine of that arch: and the velocity will not be proportional to this force, or to either the coſine or verſed ſine, but to the right ſine of the ſame arch, as will appear from what Newton has demonſtrated on this ſubject.

Into theſe, and other difficulties and obſcurities, have the writers on ſounds been betrayed, by rejecting the 47th prop. B. 2. of Newton as inconcluſive reaſoning; which propoſition is, in fact, the only true ſource from whence all the valuable diſcoveries in the doctrine of ſound are to be derived.

This demonſtration is certainly obſcurely ſtated by Newton; and by omitting a few ſteps which are neceſſary to the argument, he has given occaſion to ſome philoſophers to reject it as an argument in a circle; and which would equally tend to prove, that the pulſes are propagated according to many contrary laws.

As this propoſition therefore has been ſo ſeverely treated, and baniſhed out of the catalogue of demonſtrations, I ſhall endeavour, by propoſing it in as clear a light as I can, to reſtore it to that honour, of which, in my opinion, it has been ſo unjuſtly deprived. For this purpoſe I ſhall take the liberty of varying, in ſome degree, from the method of Newton.

3. The parts of all ſounding bodies vibrate according to the law of a cycloidal pendulum. For they may be conſidered as compoſed of an indefinite number of elaſtic fibres; but theſe fibres vibrate according to that law. *Vide Helſham, p.* 270.

4. Sounding bodies propagate their motions on all ſides *in directum*, by ſucceſſive condenſations and rarefactions, and ſucceſſive goings forward and re-

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turnings backward of the particles. *Vide prop.* 43, *B.* 2. *Newton.*

5. The pulſes are thoſe parts of the air which vibrate backwards and forwards; and which, by going forward, ſtrike (*pulſant*) againſt obſtacles. The latitude of a pulſe is the rectilineal ſpace through which the motion of the air is propagated during one vibration of the ſounding body.

All parts vibrat
backwards & forw

6. All pulſes move equally faſt. This is proved by experiment; and it is found that they deſcribe 1070 Paris feet, or 1142 London feet in a ſecond, whether the ſound be loud or low, grave or acute.

7. Prob. To determine the latitude of a pulſe. Divide the ſpace which the pulſe deſcribes in a given time (6) by the number of vibrations performed in the ſame time by the ſounding body,

(*cor.* 1, *prop.* 24, *Smith's Harmonics*); the quotient is the latitude.

Monſ. Sauveur, by ſome experiments on organ-pipes, found that a body, which gives the graveſt harmonic ſound, vibrates 12 times and an half in a ſecond, and that the ſhrilleſt ſounding body vibrates 51100 times in a ſecond. At a medium, let us take the body which gives what Sauveur calls his *fixed ſound*; it performs 100 vibrations in a ſecond, and in the ſame time the pulſes deſcribe 1070 Pariſian feet; therefore the ſpace deſcribed by the pulſes whilſt the body vibrates once, that is, the latitude or interval of the pulſe, will be 10,7 feet.

8. Prob. To find the proportion which the greateſt ſpace, through which the particles of the air vibrate, bears to the radius of a circle, whoſe perimeter is equal to the latitude of the pulſe.

During the firſt half of the progreſs of the elaſtic fibre, or ſounding body, it is continually getting nearer to the next particle; and during the latter half of its progreſs, that particle is getting farther from the fibre, and theſe portions of time are equal (*Helſham*); therefore we may conclude, that at the end of the progreſs of the fibre, the firſt particle of air will be nearly as far diſtant from the fibre as when it began to move: and in the ſame manner we may infer, that all the particles vibrate through ſpaces nearly equal to that run over by the fibre.

Now Monſ. Sauveur (*Acad. Science*, *an.* 1700, *p.* 141) has found by experiment, that the middle point of a chord which produces his *fixed ſound*, and whoſe diameter is $\frac{1}{6}$th of a line, runs over in its ſmalleſt ſenſible vibrations $\frac{1}{18}$th of a line, and in its greateſt vibrations 72 times that ſpace; that is, 72 × $\frac{1}{18}$th of a line, or 4 lines, that is, ⅓d of an inch.

The latitude of the pulſes of this fixed ſound is 10,7 feet (7); and ſince the circumference of a circle is to its radius as 710 is to 113, the greateſt ſpace deſcribed by the particles will be to the radius of a circle, whoſe periphery is equal to the latitude of the pulſe as ⅓d of an inch is to 1,7029 feet, or 20,4348 inches, that is, as 1 to 61,3044.

If the length of the ſtring be increaſed or diminiſhed in any proportion, *cæteris paribus*, the greateſt ſpace deſcribed by its middle point will vary in the ſame proportion. For the inflecting force is to the tending force, as the diſtance of the ſtring from the middle point of vibration to half the length of the ſtring (*ſee Helſham and Martin*); and therefore the inflecting and tending forces being given, the ſtring will vibrate through ſpaces proportional to its length; but the latitude of the pulſe is inverſely as the number of vibrations performed by the ſtring in a given time, (7) that is, directly as the time of one vibration, or directly as the length of the ſtring (*prop.* 24. *cor.* 7. *Smith's Harmonics*); therefore the greateſt ſpace thro' which the middle point of the ſtring vibrates, will vary in the direct ratio of the latitude of the pulſe, or of the radius of a circle whoſe circumference is equal to the latitude, that is, it will be to that radius as 1 to 61,3044.

9. As I have ventured, in ſome particulars, to vary from the method which Newton has purſued in inveſtigating the law of the pulſes, it may not, perhaps, be unneceſſary to lay before the reader a ſhort view of the ſteps by which I have endeavoured to illuſtrate his reaſoning; if, indeed, there be any

advantage in the preſent form, either as to perſpicuity or ſtrength.

In (10) it is to be proved, if the particles of air be agitated by any force whatſoever, ſo as ſucceſſively to ſet out from their proper places with a motion varying according to the law of a cycloidal pendulum, that the comparative elaſtic forces, ariſing from their mutual action, by which they will be afterwards agitated, will cauſe them to continue that motion undiſturbed to the end of their vibration. This was a neceſſary preliminary, for although it ſhould be proved, that the particles of the pulſes were at firſt ſucceſſively agitated, according to the law of a pendulum, yet air might have been a medium of ſuch a conſtitution, as immediately to diſturb that law. From hence it follows, if we can by any means prove, that the particles of air in the pulſes ſucceſſively ſet out from their proper places, according to the law of a pendulum, that they will finiſh, undiſturbed, their vibrations according to the ſame law. Now it is the buſineſs of the two following ſections, *viz.* the 11th and 12th, to demonſtrate, that the particles do really ſet out from their proper places in this manner.

In (11) the comparative elaſtic force is tobe inveſtigated with which every particle, after it has been ſet in motion according to the law of a pendulum, is agitated in the different points of its progreſs; and thence the force with which it ſets out from its proper place, before the ſubſequent particle has yet begun to move; which force is proved to be directly as its diſtance from the middle point of vibration, provided the preceding particle has made its approach to it with a motion varying according to the law already mentioned.

Now ſince this force, with which the particle begins to move, ariſes from the preceding particle's approaching it according to this law, and ſince the particle of an elaſtic fibre approaches the adjacent particle of air in this manner, it is ſhewn in (12) that this adjacent particle will ſet out with a motion varying according to the law of a pendulum. And that ſince this adjacent particle approaches the ſubſequent particle, according to the law of a pendulum, this ſubſequent particle muſt ſet out from its proper place in the ſame manner; and ſo on in ſucceſſion. And therefore, that, ſince all the particles in the medium ſucceſſively ſet out from their proper places in this manner, the motion will continue to be propagated, undiſturbed, during the whole vibration. This ſhort ſummary of the argument being premiſed, we proceed to ſtate the demonſtration itſelf.

10. If the particles of the aerial pulſes, during any part of their vibration, be ſucceſſively agitated, according to the law of a cycloidal pendulum, the comparative elaſtic forces ariſing from their mutual action, by which they will afterwards be agitated, will be ſuch as will cauſe the particles to continue that motion, according to the ſame law, to the end of their vibration.

(*Fig.* 1.) Let AB, BC, CD, &c. denote the equal diſtances of the ſucceſſive pulſes; ABC the direction of the motion of the pulſes propagated from A towards B; E, F, G three phyſical points of the quieſcent medium, ſituated in the right line AC at equal diſtances from each other; E*e*, F*f*, G*g* the very ſmall equal ſpaces through which theſe particles vibrate; ε, φ, γ any intermediate places of theſe points.

contemporaneous

Draw the right line PS equal to E*e*, biſect it in O, and from the centre O with the radius OP deſcribe the circle SIP*h*. Let the whole time of the vibration of a particle and its parts be denoted by the circumference of this circle and its proportional parts. And ſince the particles are ſuppoſed to be at firſt agitated according to the law of a cycloidal pendulum, if at any time PH, or PHS*h*, the perpendicular HL or *hl* be let fall on PS, and if E*e* be taken equal to PL or P*l*, the particle E ſhall be found in *e*. Thus will the particle E perform its vibrations according to the law of a cycloidal pendulum. *Prop.* 52. *B.* 1. *Principia.*

It should be proved
be possible

Let us ſuppoſe now, that the particles have been ſucceſſively agitated, according to this law, for a certain time, by any cauſe whatſoever, and let us examine what will be the comparative elaſtic forces, ariſing from their mutual action, by which they will afterwards continue to be agitated.

In the circumference PHS*b* take the equal arches HI, IK in the ſame ratio to the whole circumference which the equal right lines EF, FG have to BC the whole interval of the pulſes; and let fall the perpendiculars HL, IM, KN. Since the points E, F, G are ſucceſſively agitated in the ſame manner, and perform their entire vibrations of progreſs and regreſs while the pulſe is propagated from B to C, if PH be the time from the beginning of the motion of E, PI will be the time from the beginning of the motion of F, and PK the time from the beginning of the motion of G; and therefore $E\varepsilon$, $F\varphi$, $G\gamma$ will be reſpectively equal to PL, PM, PN in the progreſs of the particles. Whence $\varepsilon\varphi$ or $EF + F\varphi - E\varepsilon$ is equal to $EF - LM$. But $\varepsilon\varphi$ is the expanſion of EF in the place $\varepsilon\psi$, and therefore this expanſion is to its mean expanſion as EF — LM to EF. But LM is to IH as IM is to OP, and IH is to EF as the circumference PHS*b* is to BC; that is, as OP is to V, if V be the radius of a circle whoſe circumference is BC; therefore, *ex æquo*, LM is to EF as IM is to V; and therefore the expanſion of EF in the place εφ is to its mean expanſion, as V — IM is to V; and the elaſtic force exiſting between the phyſical points E and F is to the mean elaſtic force as $\frac{1}{V-IM}$ is to $\frac{1}{V}$ (*Cotes Pneum. Lect.* 9.) By the ſame argument, the elaſtic force exiſting between the phyſical points F and G is to the mean elaſtic force as $\frac{1}{V-KN}$ is to $\frac{1}{V}$; and the difference between theſe forces is to the mean elaſtic force as

$\frac{IM-KN}{V^2-V.IM-V.KN+IM.KN}$ is to $\frac{1}{V}$; that is, as $\frac{IM-KN}{V^2}$ is to $\frac{1}{V}$, or as IM — KN is to V;

if only (upon account of the very narrow limits of the vibration) we ſuppoſe IM and KN to be indefinitely leſs than V. Wherefore, ſince V is given, the difference of the forces is as IM — KN, or as HL — IM (becauſe KH is biſected in I); that is, (becauſe HL — IM is to IH as OM is to OI or OP, and IH and OP are given quantities) as OM; that is, if F*f* be biſected in Ω as Ωφ.

In the ſame manner it may be ſhewn, that if PHS*h* be the time from the beginning of the motion of E, PHS*i* will be the time from the beginning of the motion of F, and PHS*k* the time from the beginning of the motion of G; and that the expanſion of EF in the place $\varepsilon\varphi$ is to its mean expanſion as EF + Fφ — Eε, or as EF + *lm* is to EF, or as V + *hl* is to V in its regreſs; and its elaſtic force to the mean elaſtic force as $\frac{1}{V+hl}$ is to $\frac{1}{V}$; and that the difference of the elaſtic forces exiſting between E and F, and between F and G is to the mean elaſtic force as $kn - im$ is to V; that is, directly as $\Omega\varphi$.

But this difference of the elaſtic forces, exiſting between E and F, and between F and G, is the comparative elaſtic force by which the phyſical point φ is agitated; and therefore the comparative accelerating force, by which every phyſical point in the medium will continue to be agitated both in progreſs and regreſs, will be directly as its diſtance from the middle point of its vibration; and conſequently, will be ſuch as will cauſe the particles to continue their motion, undiſturbed, according to the law of a cycloidal pendulum. *Prop.* 38. *l.* 1. *Newton.*

Newton rejects the quantity $\mp V \times \overline{IM \mp KN} + IM \times KN$ on ſuppoſition that IM and KN are indefinitely leſs than V.

Now although this may be a reaſonable hypotheſis, yet, that this quantity may be ſafely rejected, will, I think, appear in a more ſatisfactory manner from the following conſiderations derived from experiment: PS, in its greateſt poſſible ſtate, is to V as 1 is to 61,3044 (8); and therefore IM or KN, in its greateſt poſſible ſtate, (that is, when the vibrations of the body are as great as poſſible, and the particle in the middle point of its vibration) is to V as 1 is to 122,6. Hence $V^2 = 15030{,}76$, — $V \times \overline{IM + KN} = 245{,}2$ and $IM \times KN = 1$; therefore V^2 is to $V^2 - V \times \overline{IM + KN} + \overline{IM \times KN}$ as 15030,76 is to 14786,56; that is, as 61 is to 60 nearly.

Hence it appears, that the greateſt poſſible error in the accelerating force, in the middle point, is the $\frac{1}{61}$th part of the whole. In other points it is much leſs; and in the extreme points the error entirely vaniſhes.

We ſhould alſo obſerve, that the ordinary ſounds we hear are not produced by the greateſt poſſible vibrations of which the ſounding body is capable; and that in general IM and KN are nearly evaneſcent with reſpect to V. And very probably the diſagreeable ſenſations we feel in very loud ſounds, ariſe not only from IM or KN bearing a ſenſible proportion to V, by which means the cycloidal law of the pulſes may be in ſome meaſure diſturbed, but alſo from the very law of the motion of the ſounding body itſelf being diſturbed. For, the proof of this law's being obſerved by an elaſtic fibre is founded on the hypotheſis that the ſpace, thro' which it vibrates, is indefinitely little with reſpect to the length of the ſtring. *See Smith's Harmonics, p.* 237, *Helſham, p.* 270.

11. If a particle of the medium be agitated, according to the law of a cy-

cloidal pendulum, the comparative elaſtic force, acting on the adjacent particle, from the inſtant in which it begins to move, will be ſuch as will cauſe it to continue its motion according to the ſame law.

For let us ſuppoſe, that three particles of the medium had continued to move for times denoted by the arches PK, PI, PH, the comparative elaſtic force, acting on the ſecond during the time of its motion, would have been denoted by HL — IM, that is, would have been directly as MO (10). And if this time be diminiſhed till I becomes coincident with P, that is, if you take the particles in that ſtate when the ſecond is juſt beginning to move, and before the third particle has yet been ſet in motion; then the point M will fall on P, and MO become PO; that is, the comparative elaſtic force of the ſecond particle, at the inſtant in which it begins to move, will be to the force with which it is agitated in any other moment of time, before the ſubſequent particle has yet been ſet in motion, directly as its diſtance from the middle point of vibration. Now this comparative elaſtic force, with which the ſecond particle is agitated in the very moment in which it begins to move, ariſes from the preceding particle's approaching it according to the law of a pendulum; and therefore, if the preceding particle approaches it in this manner, the force by which it will be agitated, in the very moment it begins to move, will be exactly ſuch as ſhould take place in order to move it according to the law of a pendulum. It therefore ſets out according to that law, and conſequently the ſubſequent elaſtic forces, generated in every ſucceſſive moment, will alſo continue to be of the juſt magnitude which ſhould take place, in order to produce ſuch a motion.

12. The pulſes of the air are propagated from ſounding bodies, according to the law of a cycloidal pendulum. (*Fig.* 1.) The point E of any elaſtic fibre producing a ſound, may be conſidered as a particle of air vibrating according to the law of a pendulum (3). This point E will therefore move according to this law for a certain time, denoted by the arch IH, before the ſecond particle begins to move; for ſound is propagated in time through the ſucceſſive particles of air (6). Now from that inſtant, the comparative elaſtic force which agitates F, is (11) directly as its diſtance from the middle point of vibration. F therefore ſets out with a motion according to the law of a pendulum: and therefore the comparative elaſtic force by which it will be agitated until G begins to move, will continue that law (11). Conſequently F will approach G in the ſame manner as E approached F, and the

comparative elaſtic force of G, from the inſtant in which it begins to move, will be directly as its diſtance from the middle point of vibration; and ſo on in ſucceſſion. Therefore all the particles of air in the pulſes ſucceſſively ſet out from their proper places according to the law of a pendulum, and therefore (10) will finiſh their entire vibrations according to the ſame law.

Here, by the way, we may obſerve the invalidity of the objection which Gabriel Cramer has advanced againſt this demonſtration of Newton *(ſee the Commentary of Le Seur and Jacquier)*; he attempts to ſhew, that by a mode of reaſoning exactly ſimilar, he can prove the pulſes of the air to be agitated according to a law different from Newton's; as for inſtance, according to the law of a heavy body aſcending and deſcending. And the method which he purſues is as follows: on ſuppoſition that the particles were at firſt agitated in this manner, he proceeds to inveſtigate what would be the comparative accelerating force of each particle in the medium during its progreſs and regreſs; and this he finds would be conſtant. Whence he concludes that he has, as fully as Newton, proved the truth of his propoſition. But in every ſyntheſis we are to proceed from ſome known eſtabliſhed truth: this we have done in (12), it being the known law of ſounding bodies, that they vibrate according to the law of a pendulum. But Cramer has not ſet out from any known truth; for there is no ſonorous body, we know of, which vibrates according to the law laid down by him.

Cor. 1. The number of pulſes propagated is the ſame with the number of vibrations of the tremulous body, nor is it multiplied in their progreſs: becauſe the little phyſical line $\epsilon\gamma$, (*fig.* 1) as ſoon as it returns to its proper place, will there quieſce; for its velocity, which is denoted by the ſine IM, then vaniſhes, and its denſity becomes the ſame with that of the ambient medium. This line therefore will no longer move, unleſs it be again driven forwards by the impulſe of the ſounding body, or of the pulſes propagated from it.

It has been objected to the Newtonian ſyſtem, that if the firſt pulſe of ſound be driven by that which immediately follows, and that by the ſucceeding, and ſo on, it muſt then happen, that the more numerous the pulſes, the farther will the ſound be propagated; ſo that a ſtring which vibrates the longeſt will be heard at the greateſt diſtance, which is contrary to known experience. (*See Encyclopædia Britann. article Acouſticks*). Now this objection proceeds on the ſuppoſition, that the particles of the pulſe next the body are driven forward a certain ſpace by its firſt vibration, where they would continue until driven forward again by the next; which is directly contrary to Newton's ſyſtem; for he has demonſtrated in this corollary, that the particles return in ſucceſſion to their proper places, where their vibrations end, and where they would continue at reſt, unleſs ſet in motion by a new impulſe of the ſounding body; while, at the ſame time, the condenſation is uniformly carried forward through the medium, ſome particles being in their progreſs whilſt others are in their regreſs. So that every ſucceſſive vibration of the body ſets in motion the very ſame ſeries of particles with the preceding; which motion is propagated ſucceſſively, through the ſubſequent particles, to the ſame diſtance, whether the vibrations of the ſonorous body be repeated or not.

Cor. 2. In the extreme points of the little ſpace through which the particle vibrates, the expanſion of the air is in its natural ſtate; for the expanſion of the phyſical line is to its natural expanſion as $V \mp IM$ is to V; but IM is then equal to nothing. In the middle point of the progreſs the condenſation is greateſt; for IM is then greateſt, and conſequently the expanſion $V - IM$ leaſt. In the middle of the regreſs, the rarefaction is greateſt; for im, and conſequently $V + im$ is then greateſt.

13. To find the velocity of the pulſes, the denſity and elaſtic force of the medium being given.

This is the 49th prop. B. 2. Newton, in which he ſhews, that whilſt a pendulum, whoſe length is equal to the height of the homogeneous atmoſphere, vibrates once forwards and backwards, the pulſes will deſcribe a ſpace equal to the periphery of a circle deſcribed with that altitude as its radius.

Cor. 1. He thence ſhews, that the velocity of the pulſes is equal to that which a heavy body would acquire in falling down half the altitude of that homogeneous atmoſphere; and therefore, that all pulſes move equally faſt, whatever be the magnitude of PS, or the time of its being deſcribed; that is, whether the tone be loud or low, grave or acute. *See Hales de Sonis*, § 49.

Cor. 2. And alſo, that the velocity of the pulſes is in a ratio compounded of the direct ſubduplicate ratio of the elaſtic force of the medium, and the inverſe ſubduplicate of its denſity. Hence ſounds move ſomewhat faſter in ſummer than in winter. *See Hales de Sonis*, *p*. 141.

14. The ſtrength of a tone is as the moment of the particles of air. The

moment of theſe particles, (the medium being given) is as their velocity: and the velocity of theſe particles is as the velocity of the ſtring which ſets them in motion (12). The velocities of two different ſtrings are equal when the ſpaces which they deſcribe in their vibrations are to each other as the times of theſe vibrations: therefore, two different tones are of equal ſtrength, when the ſpaces, through which the ſtrings producing them vibrate, are directly as the times of their vibration.

15. Let the ſtrength of the tones of the two ſtrings AB, CD, which differ in tenſion only (*fig*. 25, 26,) be equal. Quere the ratio of the inflecting forces F and *f*. From the hypotheſis of the equality of the ſtrength of the tones it follows (14), that the ſpace GE muſt be to the ſpace HF as $f^{\frac{1}{2}}$ to $F^{\frac{1}{2}}$, (*Smith's Harm. Prop*. 24. *Cor*. 4.) Now the forces inflecting AB, CD through the equal ſpaces GE, HP are to each other as the

tending forces, that is, as F to f, (*Malcolm's Treatiſe on Muſic*, *p.* 52.) But the force inflecting CD through HP is to the force inflecting it through HF as HP or GE to HF, (*ib.* *p.* 47.) that is, by the hyp. as $f^{\frac{1}{2}}$ to $F^{\frac{1}{2}}$. Therefore *ex æquo*, the forces inflecting AB and CD, when the tones are equally ſtrong, are to each other as $F \times f^{\frac{1}{2}}$ to $f \times F^{\frac{1}{2}}$, or as $F^{\frac{1}{2}}$ to $f^{\frac{1}{2}}$. That is, the forces neceſſary to produce tones of equal ſtrength in various ſtrings which differ only in tenſion, are to each other in the ſubduplicate ratio of the tending forces, that is, inverſely as the time of one vibration, or directly as the number of vibrations performed in a given time. Thus, if CD be the acute octave to AB, its tending force will be quadruple that of AB, (*Malcolm's Treatiſe on Muſic*, *p.* 53;) and therefore to produce tones of equal ſtrength in theſe ſtrings, the force impelling CD muſt be double that impelling AB; and ſo in other caſes.

Suppoſe now that the ſtrings AB, CD (*fig.* 26, 27,) differ in length only. The force inflecting AB through GE is to the tending force, which is given, as GE to AG; and this tending force is to the force inflecting CD through the ſpace HP equal to GE, as HD to HP. Therefore, *ex æquo*, the forces inflecting AB and CD through the equal ſpaces GE and HP are to each other as HD to AG, or as CD to AB. But the force inflecting CD through HP is to the force inflecting it through HF, as HP or GE to HF, that is, becauſe theſe ſpaces are as the times (14), as AB to CD. Therefore, *ex æquo*, the forces inflecting AB and CD, when the tones are equally ſtrong, are to each other in a ratio of equality. Hence we ſhould ſuppoſe, that in this caſe, an equal number of equal impulſes would generate equally powerful tones in theſe ſtrings. But we are to obſerve, that the longer the ſtring the greater, *cæteris paribus*, is the ſpace through which a given force inflects it (*Malcolm*); and therefore whatever diminution is produced in the ſpaces through which the ſtrings move in their ſucceſſive vibrations, ariſing either from the want of perfect elaſticity in the ſtrings, or from the reſiſtance of the air, this diminution will bear a greater proportion to the leſs ſpace, through which the ſhorter ſtring vibrates. And this is confirmed by experience; for we find that the duration of the tone and motion of the whole ſtring exceeds that of any of its ſubordinate parts. Therefore, after a given interval of time, a greater quantity of motion will remain in the longer ſtring; and conſequently after the ſucceſſive equal impulſes have been made, a greater degree of motion will ſtill ſubſiſt in it. That is, a given number of equal impulſes being made on various ſtrings differing in length only, a ſtronger ſound will be produced in that which is the longer.

SECT. II.

OF THE DECAY OF SOUND.

16. IN accounting for the decay of ſound, according to the diſtance from the ſounding body, we are told, that whatever be the force wherewith the ſounding body acts on the firſt ſpherical ſhell of air, with the very ſame force does that ſhell act on the ſecond, and that again upon the third, and ſo on; ſo that the force condenſing the air in the ſeveral ſhells is given; conſequently, the condenſations which it produces in theſe ſhells muſt be inverſely as the reſiſtance it meets with: but the reſiſtances are as the ſhells, and therefore, ſince theſe increaſe continually in the ſame proportion with the ſquares of the

diſtances from the centre, their condenſations muſt decreaſe in the ſame manner. And hence it is, ſay the writers on theſe ſubjects, that ſounds grow leſs and leſs audible the farther they go from the ſounding body. *See Merſennus*, *Helſham*, *&c.*

But this reaſoning is founded on the hypotheſis, that a body in motion communicates to another, which it impells, a quantity of motion equal to its own. This however is not always true; and it is, particularly in this caſe, contradicted by their own theory; for as theſe ſhells are elaſtic, and therefore as motion is communicated from a ſmaller elaſtic body to a greater, there ſhould be an increaſe of motion; and conſequently, the compreſſing forces in the ſeveral ſhells are not of the ſame magnitude. And further, the very ſame mode of arguing may be applied to overturn their theory of the increaſe of ſound in ſpeaking-trumpets; where, in the ſame manner, we may ſay, that the force with which the firſt cylinder of air acts on the ſecond, is equal to that with which the ſecond acts on the third, and ſo on: therefore the force, with which the ſeveral cylinders of air act on each other, is given, and conſequently the condenſations will be inverſely as the reſiſtances: but theſe reſiſtances, in the beſt contrived tubes, increaſe in geometrical progreſſion; therefore the condenſations, and conſequently the ſtrength of the ſound, ſhould decreaſe in the ſame proportion.

But even ſuppoſing that we are to conſider the motion of ſound as communicated in this manner from one elaſtic ſhell of air to another, we ſhall find, that the decay of ſound would not follow the law which has been laid down, *viz.* the inverſe duplicate of the diſtance. For let a ſeries of elaſtic bodies be repreſented by A, B, C, &c. motion being communicated from A to B, and from B to C, and ſo on. Let the velocities of thoſe bodies be denoted by a, b, c, &c. the general ſeries repreſenting the quantities of motion of each body in ſucceſſion, according to the law of colliſions, will be Aa. $\frac{2AaB}{A+B} = Bb \,.\, \frac{2BbC}{B+C} = Cc$, &c. (*Helſham*, *p.* 361), where we ſee, that the quantity of motion in every preceding ſhell is leſs than that of the ſucceeding, in the ratio of the ſum of the two ſhells to twice the greater; and the ſeries exhibiting the velocities will be, $a \,.\, \frac{2Aa}{A+B} = b \,.\, \frac{2Bb}{B+C} = c$, &c. whence we ſee, that the velocity of every preceding ſhell is greater than that of the ſucceeding, in the proportion of the ſum of the two ſhells to twice the leſs. Therefore in this caſe, ſince the radii of the ſhells increaſe as the natural numbers 1, 2, 3, 4, &c. the ſhells themſelves will be as 1, 4, 9, 16,

&c. and the velocities, according to the above ſeries, will be had by multiplying into each other the terms of the following ſeries, $1, \frac{2}{5}, \frac{8}{13}, \frac{18}{25}$, &c. therefore the velocity, and conſequently the force of the ſound, decreaſes through theſe ſhells in the following proportion, $1 \cdot \frac{2}{5} \cdot \frac{16}{65} \cdot \frac{288}{1625}$, &c. that is, as $1 \cdot \frac{1}{2\frac{1}{2}} \cdot \frac{1}{4\frac{1}{16}} \cdot \frac{1}{5\frac{185}{288}}$, &c. whereas agreeable to the received theory, it ſhould decreaſe in the following proportion, $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}$, &c.

It might be expected that I ſhould not here paſs over, unobſerved, the experiment adduced by Sir Iſaac Newton in favour of the received doctrine. Let there be two ſonorous bodies, ſays he, whoſe tones are equally ſtrong, the one in the open air, the other included in a receiver parly exhauſted. If their diſtances be in a ſubduplicate ratio of the denſities of the air in which they vibrate, their tones will ſtrike the ear with equal force. For inſtance, if only one hundredth part of the whole air remain in the receiver, the ſound ſhould be one hundred times more languid; and therefore ſhould be heard no more diſtinctly, than if a perſon in the open air ſhould recede ten times the diſtance from the ſounding body. Now it appears, that the ſounds in both caſes are equally ſtrong. Therefore, the ſounds muſt have decreaſed in the inverſe duplicate ratio of the diſtance. (*See Principia, B.* 2. *prop.* 47. *firſt edition; or Hales de Sonis, p.* 41). This experiment is, for ſeveral reaſons, inconcluſive; it is next to impoſſible to contrive it ſo as that the original ſounds ſhall be exactly in the direct ratio of the denſities of the air; and even admitting this were poſſible, yet, no muſician could pretend to determine, whether two ſounds which approach nearly to an equal degree of ſtrength are, in fact, accurately ſo or not. But in truth

Newton himſelf ſeems to have been conſcious of the imperfection of this experiment, from his omitting it in the ſubſequent editions of his Principia.

The principal cauſe of the decay of ſound ſeems to me to be the want of perfect elaſticity in the air, whence it ariſes, that every ſubſequent particle has not the entire motion of the preceding particle communicated to it, as in the caſe of equal and perfectly elaſtic bodies; whence the farther the motion is propagated, the more will the velocity with which the particles move be diminiſhed, and therefore the leſs the ſpace PS (*fig.* 1), thro' which they perform their vibrations; and the leſs the *ſinus totus* IM. But in all vibrations of the ſounding body, the latitude of the pulſes, and conſequently the quantity V, remains the ſame; therefore the farther the motion is propagated the leſs will be the quantity $\frac{1}{V-IM}$ which expreſſes the condenſation; therefore the farther the pulſe is propagated, the more is the denſity, and conſequently the impulſe on the drum of the ear, diminiſhed.

And that the defect of perfect elaſticity in the air is a principal cauſe of the decay of ſound appears from this, that ſounds are perceived more diſtinctly when north, north-eaſt, and eaſterly winds prevail, as Derham and Kircher have obſerved, (*ſee Phil. Tranſ. an.* 1705); at which time the air is dry, (*ſee Du Luc*, *V.* 2. *p.* 203), and conſequently more elaſtic, for vapours diffuſed through the atmoſphere, unleſs dilated by intenſe heat, diminiſh the ſpring of the air, (*ibid. p.* 194). And therefore the ſpaces deſcribed by the vibrating particles do not decreaſe ſo rapidly, though the barometer be at that time higheſt, (*ſee Du Luc*, *V.* 2. *p.* 203), and conſequently the maſs of air, to be moved, greateſt.

We may alſo deduce the ſame inference from obſerving, that the tone of a ſounding body, at a given diſtance, gradually languiſhes; which ariſes from this, that in every ſucceſſive vibration, the ſpace through which the ſtring, and conſequently every particle of air, vibrates gradually diminiſhes. And therefore we may attribute the decay of ſound at different diſtances to the ſame cauſe; that is, to a diminution of the ſpace through which the particles vibrate, ariſing, in a great meaſure, from their want of perfect elaſticity.

SECT. III.

Of SPEAKING-TRUMPETS.

17. WRITERS in general account for the augmentation of ſound in ſpeaking-trumpets by the communication of motion from a ſmaller elaſtic body to a greater; in which caſe there is always an increaſe of motion, as appears from (16). And therefore, as this happens in the ſucceſſive plates of air through the whole length of the tube, the laſt plate muſt emerge from the mouth of the tube with a very conſiderable increaſe of motion.

From this reaſoning they have been led to conſider the beſt form for ſuch tubes, and have concluded that to be the beſt, which is generated by the revolution of the logarithmic curve round its axis, as in a tube of this form the elaſtic bodies will increaſe in ſuch a manner as moſt to increaſe the quantity of motion. (*See Helſham, p.* 75, *&c.*) But though there may be, in theſe caſes, an increaſe of motion, there certainly is a decreaſe of velocity (16), therefore the ſpace PS (p. 46), and conſequently IM will diminiſh (*fig.* 1); therefore the condenſation cannot poſſibly derive any augmentation from the increaſe of the whole motion merely. Though the laſt plate of air emerges from the mouth of the tube with a conſiderable increaſe of motion, yet, as its velocity will have decreaſed, the denſity, and therefore the force of the pulſe, will have decreaſed alſo. The ear in all caſes is affected by a cylinder of air of a given baſe; and therefore in eſtimating the magnitude of the ſtroke, we are not to conſider the quantity of motion which may ſubſiſt in the whole body of the atmoſphere, but what ſubſiſts in that part of it only which ſtrikes on the tympanum. And therefore, though the motion of the whole maſs may be increaſed, yet, the motion of that particular cylinder which produces the impulſe on the ear, may be, and really is, diminiſhed (16); even according to their own theory.

18. Martin, V. 2. p. 247, and many others, (*ſee Encyclopædia Brittan. article Acouſticks*,) derive the augmentation of the ſound principally from its confinement; and calculate its increaſe in the following manner: the intenſity of the naked voice is to that with which it iſſues from the mouth of the tube, as the ſegment of the ſpherical ſurface, deſcribed with the length of the tube as its radius, and whoſe baſe is the orifice of the tube, to that entire ſpheri-

cal ſurface. Becauſe, the voice, by receding in every direction from the mouth, is dilated through that entire ſurface, but at its iſſuing from the tube, it is all confined to the above-mentioned ſegment; and the intenſities of the voice will be reciprocally as the maſſes of air to be moved. But it is evident, that the confinement of the voice can have little effect in increaſing the ſtrength of the ſound, as this ſtrength depends on the velocity with which the particles move. Were this reaſoning concluſive, the voice ſhould iſſue through the ſmalleſt poſſible orifice; cylindrical tubes would be preferable to any that increaſed in diameter; and the leſs the diameter, the greater would be the effect of the inſtrument, becauſe the plate or maſs of air to be moved would, in that caſe, be leſs, and conſequently the effect of the voice the greater; all which is contradicted by experience.

19. The cauſe of the increaſe of ſound in theſe tubes muſt therefore be derived from ſome other principles: and amongſt theſe we ſhall probably find, that what the ingenious Kircher has ſuggeſted in his Phonurgia is the moſt deſerving of our attention. He tells us that "the augmentation of the ſound depends on its reflection from the tremulous ſides of the tube; which reflections, conſpiring in propagating the pulſes in the ſame direction, muſt increaſe its intenſity." Newton alſo ſeems to have conſidered this as the principal cauſe, in the ſcholium of prop. 50. B. 2. Princip. when he ſays, "we hence ſee why ſounds are ſo much increaſed in ſtentorophonic tubes, for every reciprocal motion is, in each return, increaſed by the generating cauſe."

Farther, when we ſpeak in the open air, the effect on the tympanum of a diſtant auditor is produced merely by a ſingle pulſe. But when we uſe a tube, all the pulſes propagated from the mouth, except thoſe in the direction of the axis, ſtrike againſt the ſides of the tube, and every point of impulſe becoming a new centre, from whence the pulſes are propagated in all directions, a pulſe will arrive at the ear from each of thoſe points; thus, by the uſe of a tube, a greater number of pulſes are propagated to the ear, and conſequently the ſound increaſed. The confinement too of the voice may have ſome effect, tho' not ſuch as is aſcribed to it; for the condenſed pulſes produced by the naked voice, freely expand every way; but in tubes, the lateral expanſion being diminiſhed, the direct expanſion will be increaſed, and conſequently the velocity of the particles, and the intenſity of the ſound. The ſubſtance alſo of the tube has its effect; for it is found by experiment, that the more elaſtic the ſubſtance of the tube, and the more porous, and conſequently the more ſuſceptible it is of theſe tremulous motions, the ſtronger is the ſound. Of different rooms of the ſame form and ſize, we find that ſome deaden the voice of the ſpeaker, whilſt in others we perceive it diſtinct and forcible. This can only ariſe from the different degrees of elaſticity, and conſequently of vibration in the walls of theſe chambers, by which the pulſes of the air are thus variouſly affected.

If the tube be laid on any non-elaſtic ſubſtance, it deadens the ſound, becauſe it prevents the vibratory motion of the parts. The ſound is increaſed in ſpeaking-trumpets, if the tube be ſuſpended in the air; becauſe the agitations are then carried on without interruption. Theſe tubes ſhould increaſe in diameter from the mouth-piece, becauſe the parts, vibrating in directions perpendicular to the ſurface, will conſpire in impelling forward the particles of air, and conſequently, by increaſing their velocity, will increaſe the intenſity of the ſound: and the ſurface alſo increaſing, the number of points of impulſe and of new propagations will increaſe proportionally. The ſeveral cauſes, therefore, of the increaſe of ſound in theſe tubes ſeem to be, the diminution of the lateral, and conſequently the increaſe of the direct expanſion and velocity of the included air. Secondly, the increaſe of the number of pulſes, by increaſing the points of new propagation. And thirdly, the reflections of the pulſes from the tremulous ſides of the tube, which impel the particles of air forward, and thus increaſe their velocity.

Sounds in paſſing over a rough ſurface are not varied as to *velocity*, for that depends merely on the ſtate of the air as to denſity and elaſticity (13). But they decay in *ſtrength*, becauſe the vibrations of the ſurface, being in various directions, will counteract the motion of the ſound. In diſtinguiſhing theſe effects we ſhould obſerve, that the ſtrength of the tone does not depend on the greater velocity of the *pulſes*, but of the *particles* of air which conſtitute them.

SECT. IV.

OF ECHOS.

20. THE phænomena of echos are thus accounted for by the writers on acouſticks: If ſounds in their progreſs ſtrike againſt a proper obſtacle, they will partly divaricate into the ſpaces behind the obſtacle; and partly rebounding from thence, as from a centre, they will be again propagated back in an angle equal to the angle of incidence, and by this reflection will generate a ſecond ſound, or echo. (*See Kircher's Phonurgia; Jacquier and Le Seur on Newton, B.* 2. § 324; *Martin's Phil. Brit. V.* 2. *p.* 237; *Encyclop. Brit. article Acouſticks*). Writers have, in general, reaſoned on the pulſes of air as on the

rays of light, which are reflected in an angle equal to the angle of incidence. "As to the reflection of ſound," ſays M. Lambert, "we know that it obſerves the law of all other reflections, that is to ſay, that the angle of reflection is equal to the angle of incidence; on which is founded the method of conſtructing artificial echos." *See Berlin Tranſactions, vol.* 19. *p.* 91.

But this is, no doubt, an erroneous poſition, which has probably taken its riſe from carrying the analogy between light and ſound too far. For in ſound, every point of impulſe becomes a centre from which a new ſeries of pulſes are propagated in every angle whatſoever. The principles by which theſe phænomena may be explained are few and ſimple, *viz.* that every point, againſt which the pulſes ſtrike, becomes a centre of a new ſeries of pulſes, (*prop.* 43. *B.* 2. *Newton*);

and ſecondly, that ſound deſcribes equal ſpaces in equal times (6 and 13). Therefore, when any ſound is propagated from a centre, and its pulſes ſtrike againſt a variety of obſtacles, if the ſum of the right lines drawn from that point to each of the obſtacles, and from each obſtacle to a ſecond point, be equal; then will the latter be a point in which an echo will be heard, (*fig.* 3). Thus, let A be the point from which the ſound is propagated in all directions, and let the pulſes ſtrike againſt the obſtacles C, D, E, F, G, H, I, &c. each of theſe points becomes a new centre of pulſes by the firſt principle, and therefore from each of them one ſeries of pulſes will paſs through the point B. Now if the ſeveral ſums of the right lines $\overline{AC+CB}$, $\overline{AD+DB}$, $\overline{AE+EB}$, $\overline{AG+GB}$, $\overline{AH+HB}$, $\overline{AI+IB}$, &c. be all equal to each other, it is obvious that the pulſes propagated from A to theſe points, and again from theſe points to B, will all arrive at B at the ſame inſtant, according to the ſecond principle; and therefore, if the hearer be in that point, his ear will at the ſame inſtant be ſtruck by all theſe pulſes. Now it appears from experiment, (*ſee Muſſchenbroek*, V. 2. *p.* 210), that the ear of an exerciſed muſician can only diſtinguiſh ſuch ſounds as follow one another at the rate of 9 or 10 in a ſecond, or any ſlower rate: and therefore for a diſtinct perception of the direct and reflected ſound, there ſhould intervene the interval of $\frac{1}{9}$th of a ſecond; but in this time ſound deſcribes $\frac{1142}{9}$ or 127 feet nearly (6). And therefore, unleſs the ſum of the lines drawn from each of the obſtacles to the points A and B exceeds the interval AB by 127 feet, no echo will be heard at B. Since the ſeveral ſums of the lines drawn from the obſtacles to the points A and B are of the ſame magnitude, it appears that the curve paſſing through all the points

C, D, E, F, G, H, I, &c. will be an ellipſe, *(prop. 14. B. 2. Ham. Con.)* Hence all the points of the obſtacles which produce an echo, muſt lie in the ſurface of the oblong ſphæroid, generated by the revolution of this ellipſe round its major axis.

As there may be ſeveral ſuch ſphæroids of different magnitudes, it follows, that there may be ſeveral different echos of the ſame original ſound. And as there may happen to be a greater number of reflecting points in the ſurface of an exterior ſphæroid than in that of an interior, a ſecond or a third echo may be much more powerful than the firſt, provided that the ſuperior number of reflecting points, that is, the ſuperior number of reflected pulſes propagated to the ear, be more than ſufficient to compenſate for the decay of ſound which ariſes from its being propagated through a greater ſpace. This is finely illuſtrated in the celebrated echos at the Lake of Killarney in Kerry, where the firſt return of the ſound is much inferior in ſtrength to thoſe which immediately ſucceed it.

It is not abſolutely neceſſary that the reflecting points C, D, E, F, &c. producing an echo, ſhould lie accurately in the periphery of an ellipſe; for if the ſums of the right lines $\overline{AC}+\overline{CB}$, $\overline{AD+DB}$, &c. do not differ from each other by more than 127 feet, the pulſes propagated from theſe points will not be diſtinguiſhable. However the nearer theſe ſums approach to equality, the more accurate will be the coincidence of the pulſes, and conſequently the echo will be the more diſtinct.

21. From what has been laid down it appears, that for the moſt powerful echo, the ſounding body ſhould be in

one focus of the ellipſe which is the ſection of the echoing ſphæroid, and the hearer in the other. However an echo may be heard in other ſituations, though not ſo favourably; as ſuch a number of reflected pulſes may arrive at the ſame time at the ear as may be ſufficient to excite a diſtinct perception. Thus a perſon often hears the echo of his own voice; but for this purpoſe he ſhould ſtand at leaſt 63 or 64 feet from the reflecting obſtacle, according to what has been ſaid before. At the common rate of ſpeaking, we pronounce not above three ſyllables and an half, that is, ſeven half ſyllables in a ſecond; therefore, that the echo may return juſt as ſoon as three ſyllables are expreſſed, twice the diſtance of the ſpeaker from the reflecting object muſt be equal to 1000 feet; for, as ſound deſcribes 1142 feet in a ſecond, $\frac{6}{7}$ths of that ſpace, that is, 1000 feet nearly, will be deſcribed while ſix half, or three whole ſyllables are pronounced: that is, the ſpeaker muſt ſtand near 500 feet from the obſtacle. And in general, the diſtance of the ſpeaker from the echoing ſurface, for any number of ſyllables, muſt be equal to the ſeventh part of the product of 1142 feet multiplied by that number.

In churches we never hear a diſtinct echo of the voice, but a confuſed ſound when the ſpeaker utters his words too rapidly; becauſe the greateſt difference of diſtance between the direct and reflected courſes of ſuch a number of pulſes as would produce a diſtinct ſound, is never in any church equal to 127 feet, the limit of echos.

But though the firſt reflected pulſes may produce no echo, both on account of their being too few in number, and too rapid in their return to the ear; yet it is evident, that the reflecting ſurface may be ſo formed, as that the pulſes which come to the ear after two reflections or more may, after having deſcribed 127 feet or more, arrive at the ear in ſufficient numbers, and alſo ſo nearly at the ſame inſtant, as to produce an echo, though the diſtance of the reflecting ſurface from the ear be leſs than the limit of echos. This is confirmed by a ſingular echo in a grotto on the banks of the little brook called the Dinan, about two miles from Caſtlecomber, in the county of Kilkenny. As you enter the cave, and continue ſpeaking loud, no return of the voice is perceived; but on your arriving at a certain point, which is not above 14 or 15 feet from the reflecting ſurface, a very diſtinct echo is heard. Now this echo cannot ariſe from the firſt courſe of pulſes that are reflected to the ear, becauſe the breadth of the cave is ſo ſmall, that they would return too quickly to produce a diſtinct ſenſation from that of the original ſound: it therefore is produced by thoſe pulſes, which, after having been reflected ſeveral times from one ſide of the grotto to the other, and having run over a greater ſpace than 127 feet, arrive at the ear in conſiderable numbers, and not more diſtant from each other, in point of time, than the ninth part of a ſecond.

22. Some philoſophers * of the Continent maintain, that ſounds move only in right lines from the ſounding body, in the ſame manner as light is propagated from the ſun: and that we hear it only in different directions, as we ſee light, *viz.* by reflection. This theory (which contradicts prop. 42. B. 2. Newton), they endeavour to ſupport by the obſervation, that echos are heard

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* Euler, Berlin Tranſ. an. 1763; Monſ. Lambert, ib. Monſ. Hube, Acta Erud. an. 1762, &c.

only in one direction; which, they ſay, could not be the caſe, if ſound was propagated in all directions from the ſonorous body. "The echo," ſays Monſ. Lambert, "appears to me a deciſive inſtance in this caſe. An echo is produced when ſound is reflected from an obſtacle at a ſufficient diſtance. In caſes of this nature, I have always obſerved, that the echo, ſo far from being heard all around, is heard there only, where the reflected ſound paſſes according to the laws of reflection." (*See Berlin Tranſ. an.* 1763). But this argument is inconcluſive, as we have ſhewn that the ſame effect would follow from ſounds being propagated in all manner of directions, and that moſt probably the true reaſon why the echo of the ſounding body, ſituated ſuppoſe at A (*fig.* 3), is heard only in one ſituation as at B, is not becauſe the angles of reflection at C, D, E, F, &c. are equal to the angles of incidence, (*ſee ſchol. prop.* 17. *B.* 2. *Ham. Con.*); but becauſe the ſeveral ſums of the lines $\overline{AC+CB}$, $\overline{AD+DB}$, $\overline{AE+EB}$, &c. are of the ſame magnitude (p. 60): whence it follows, that the pulſes propagated from A, and reflected from the obſtacles C, D, E, &c. will all meet in the focus B at the ſame inſtant, and conſequently produce an echo; provided the interval of time which elapſes between the perception of the original ſound and the concurrence of the pulſes at B, exceeds the ninth part of a ſecond. So that an echo's being heard only in that point where the reflected ſound paſſes according to the laws of reflection, is not a conſequence of the equality of the angles of incidence and reflection, but of the equality of the ſpaces deſcribed by ſound in equal times.

PART II.

ON MUSICAL STRINGS.

* * *

SECT. I.

OF THE MOTION OF AN ELASTIC FIBRE.

23. IF a non-elaſtic body AB receive an impulſe in C, (*fig.* 4) a twofold effect will be produced; for the body will not only go forward in the direction of the ſtroke, but the ſurface AB, will alſo be indented at E; the non-elaſtic fibre AB aſſuming the figure AEDB: which indenture will continue; the whole body in the mean time moving forward.

But if this body, and conſequently the fibre AB, were elaſtic, the inflected part AED would reſtore itſelf to its former poſition, and produce an undulation or ſerpentine motion in AB; the whole body, at the ſame time, moving forward as before.

24. The manner in which this undulatory motion is propagated, is as follows: The impulſe being made at C (*fig.* 5), at the ſame time that the whole fibre moves forward as before, the point C will relatively move faſter forward, and drawing after it equal parts of the fibre on either ſide, will form the iſoſceles triangle AED. The point D will next, relatively, be ſet in motion, becauſe it is acted on by two tending forces in the directions ED, DB; and therefore will aſcend in a line that lies between theſe directions. But becauſe AE is very nearly parallel to AB, the angle AED will be very

nearly equal to EDB, and the forces in the directions of the ſides of theſe angles are equal, being the ſame tending force of the ſtring. Therefore D will begin to aſcend with the ſame force, very nearly, with which E begins to deſcend. Now as E begins to deſcend and D to aſcend, at the ſame time, with equal forces, when E has deſcended to C, D will have aſcended to F, (*fig.* 6), nearly as far from the axis as E, the fibre aſſuming the form ACFGB. But E, by the velocity acquired in its deſcent to C, will paſs on to the other ſide of the axis, ſuppoſe to K, (*fig.* 7); F in the mean time will have deſcended, ſuppoſe, to D, and G aſcended to I; and the fibre will have aſſumed, nearly, the figure AKDIHB. The firſt wave therefore has now generated another contiguous to it, whoſe convexity lies on the oppoſite ſide of the axis; in the ſame manner, the wave DIH will generate another: and thus will an undulation be carried on throughout the entire length of the fibre, the convexities of the contiguous waves always lying on oppoſite ſides of the axis; while, as before, the whole body, collectively conſidered, is going forward in the direction of the original impulſe.

Hence it appears, that if a tended elaſtic fibre, all the parts of which are equally ſuſceptible of motion, receive an impulſe, an undulatory or ſerpentine motion will be produced therein, at the ſame time that the entire fibre is carried forward in the direction of the impulſe.

25. If now the extremities A, B be fixed, and the fibre receive an impulſe at C, an indenture will be made at C as before, and, at the ſame time, every point of the whole fibre will have a tendency forward as before; but the

two points A, B alone being fixed, theſe alone will continue at reſt, and therefore the other points of the fibre advancing forward, will cauſe the whole fibre to become concave to the axis AB. And after it has attained the utmoſt limit of its progreſs, its elaſticity will cauſe the whole fibre to return to its former poſition; and by the velocity acquired in its return, it will paſs on as far on the other ſide of the axis AB, or rectilineal poſition.

Hence will ariſe a ſlower vibration of the whole fibre by the firſt of the two effects of the impulſe mentioned in (23), while at the ſame time, as before, the minute, rapid undulation, cauſed by the other of theſe effects, is carried on throughout its entire length; the ſtring aſſuming a figure ſomewhat like that repreſented in the eighth diagram.

26. From what has been ſaid it appears, that as every wave endeavours to generate another contiguous to it, whoſe convexity ſhall lie on the oppoſite ſide of the axis; the firſt wave endeavours to generate a ſeries of equal waves. For as the time of the vibration of any wave is as its length, (*cor.* 7. *prop.* 24. *Smith's Harmonics*), if theſe waves were unequal, the times would be ſo too, and conſequently the convexities would not ſucceſſively lie on oppoſite ſides of the axis.

27. If the impulſe on any point of the ſtring be ſo great as that, during the progreſs of that point, it ſhall agitate the part of the fibre which lies between that point and the nearer end of the fibre, it is evident that the latitude of that wave, and conſequently of every ſubſequent one (26), will be double the diſtance of the point of impulſe from the neareſt extremity of the

ſtring. If half the latitude of the firſt wave be an aliquot part of the ſtring, an even and regular undulation will be propagated through it; and if it be half the whole ſtring, there will not be any undulation at all.

If half the latitude of the firſt wave be an aliquant part of the ſtring, after the motion has been propagated to the fartheſt extremity, there will be a new ſeries of leſs waves, recurring in a contrary direction. Thus, if AD be not an aliquot part of AB (*fig.* 9), after the waves have been propagated to K, KB being then ſet in motion, will excite a new ſeries of recurrent waves from B towards A: and if KB be not an aliquot part of AB, there will be a third ſeries of waves leſs than either of the former propagated back again from A to B, and ſo on *ad infinitum*. Now as muſical ſtrings are ſo very much tended, we may always ſuppoſe, that the impulſe on any point is ſufficiently ſtrong to agitate the portion of the ſtring between it and the neareſt extremity of the ſtring, during the progreſs of that point; and therefore we may conclude, that the latitude of the waves, in any ſeries, is equal to double the diſtance of the point of impulſe from the neareſt extremity of the ſtring.

28. In order to find whether this reaſoning accorded with experiment, I procured as long a ſtring as I conveniently could; and ſtraining it tight, but not ſo much as to render it muſical, I made the following remarks on its vibrations:

When I pulled it by a point near either of the extremities, and accurately examined its appearance during its vibration, I obſerved two forms of a chord, one vibrating very rapidly, while, at the ſame time, the other appeared to roll ſlowly backwards and forwards, and to croſs the former. Now thoſe ſituations in which the chord appeared well defined, were certainly the extremes of its progreſs and regreſs, where having loſt its entire motion, it muſt have continued longer than any where elſe. And as there were two appearances accurately defined, one moving ſlower, the other quicker, it evidently follows, that the chord muſt really have had two motions, one ſlower, the other quicker, according to (25). Sometimes, by ſtriking the chord at random, I have ſeen three or four of theſe apparent chords croſſing each other with various velocities (27). But when I pulled it by the middle point, there appeared but one image of the chord. When I pulled it by the middle of an aliquot part, I was not without hopes of being able to diſtinguiſh the waves of the undulation, from knowing where they would interſect the axis, and accurately obſerving it in thoſe points: but I never could diſcover any ſuch interſections. However, the experiments, to me, appear as deciſive in favour of the theory, as could be expected in a caſe where motions ſo rapid and ſubtle were the object of examination.

29. When the impulſe is made only on one point of the ſtring, and the motion thence propagated through the whole, the undulatory motion is ſuch as we have deſcribed in (24): every wave generating another contiguous to it, whoſe convexity ſhall lie on the oppoſite ſide of the axis (*ib.*) But if ſeveral impulſes be made on the middle points K, C, D, E, (*fig.* 10), of the aliquot parts AG, GH, HI, IB, at the ſame inſtant, and in the ſame direction, the ſeries of waves will be converted into a ſeries of indentures, whoſe convexities will all look the ſame way;

and, at the ſame time, each impulſe will contribute to produce a vibration of the whole ſtring (25). For theſe impulſes being equal, they will produce equal indentures in the ſtring; and as the tenſion is equal throughout the ſtring, theſe indentures will vibrate in the ſame time, and conſequently their interſections G, H, I will always continue in the ſame points: at the ſame time that the whole ſtring, collectively conſidered, has a motion forward as before; aſſuming nearly the figure repreſented in the 11th diagram.

SECT. II.

OF SYMPATHETIC TONES.

30. WE now proceed to conſider the phænomena of muſical ſtrings, and to ſhew how they are deducible from the principle which we have been hitherto endeavouring to eſtabliſh. But I am far from maintaining all that is here advanced to be perfect and rigid demonſtration; indeed the ſubtilty and intricacy of the ſubject ſeem ſcarcely to admit of it. However, the ſimplicity of the principle, as delivered in the firſt ſection of this part; the eaſe with which all the different phænomena, treated of in the following ſections, may be derived from it; and the

variety of experiments which conſpire to eſtabliſh its certainty as far as could be expected in ſuch an enquiry, will, I hope, leave little doubt of its truth.

Let the two ſtrings AB, CD (*fig.* 2) be of equal diameter, length and tenſion; and let one of them AB be ſtruck, ſo as to produce a forcible tone; the other CD will be ſo agitated by the pulſes of the air, as to produce a tone; which conſequently will be in uniſon with the former. For, the pulſes of the air vibrate in the ſame time with the ſounding body which produces them (12). Now theſe pulſes propagated towards CD, will ſtrike upon every point of it, at the ſame time and in the ſame direction: that which ſtrikes on the middle point of CD will agitate the whole ſtring; all the other impulſes will produce a double effect, both a vibration of the whole ſtring, and an indenture, according to (25). That which ſtrikes nearly on the middle of C*n* will agitate that portion, forming nearly an iſoſceles triangle whoſe baſe is C*n*; at the ſame time another equal pulſe will agitate the portion *n*D; and theſe two indentures, or bellies, being always equal, their interſection will always continue at *n*; and they will conſequently vibrate in half the time of the whole ſtring. In the ſame manner, three pulſes will ſtrike nearly, or accurately, on the middle points of three aliquot parts of CD, which will produce three interſecting indentures; and their nodes or interſections will alſo continue in the ſame points; and they will vibrate in one-third of the time of the whole vibration. And in like manner we muſt reaſon concerning all other aliquot parts.

Now of all theſe motions we are to conſider which will predominate: and that evidently muſt be the motion of the whole ſtring. For all the ſubordinate vibrations of the aliquot parts muſt comparatively be deſtroyed, becauſe they are performed in times different from that of the pulſes; but the whole ſtring vibrating exactly in the ſame time with them, every ſucceeding impulſe from the air will conſpire in direction with the motion already generated; and therefore will at length ſo far increaſe the motion of CD, as to render it ſufficiently powerful to affect the ear. And beſides, as the lengths of theſe indentures are leſs than that of the entire ſtring, the vibrations produced in them will be proportionably leſs powerful (15). As to thoſe pulſes which ſtrike on the middle points of aliquant parts of the ſtring, they can never produce any ſenſible effect. For ſuppoſe CR (*fig.* 12) an aliquant part of CD; from the extremity C take CR as often as poſſible in CD, ſuppoſe five times, there will remain the portion PD leſs than any one of theſe; the pulſes that ſtrike on the middle points of theſe equal portions produce equal effects; take from the other extremity D, the ſame number of equal parts, the pulſes that ſtrike on the middle points of theſe parts produce equal effects with the former; but as PD is leſs than QP, this ſeries of indentures will interſect and break the former; and as the two ſeries of indentures are equally powerful, they will deſtroy each other's effects. Beſides, as theſe aliquant parts are incommenſurate to AB, the times of their vibrations will be alſo incommenſurate to that of the pulſes; and therefore their vibrations will be extinguiſhed by theſe pulſes.

31. The more acute the tones of the ſtrings AB, CD (*fig.* 2), the more difficult it is to excite a tone in the quieſcent CD: for in order to produce

tones of equal ſtrength in different ſtrings, the impelling force, *cæteris paribus*, muſt be in the ſubduplicate ratio of the tending forces; that is, directly as the number of vibrations performed in a given time (15). So that to produce an octave requires a double effort; to produce a twelfth, a triple effort; a ſeventeenth major, a quintuple effort; and ſo on: conformable to what Newton tells us in the ſcholium of prop. 50. B. 2. Prin. " that the more rapid tremors are with greater difficulty excited in quieſcent objects, by the pulſes propagated from ſounding bodies."

32. If the length of CD be but a little leſs or greater than that of AB, ſtill the ſtrong vibrations of AB will cauſe it to produce a tone; becauſe if the tenſion and diameters of homogeneous chords be equal, the times of their ſingle vibrations are in the direct ratio

of their lengths (*cor.* 7. *prop.* 24. *Smith's Harmonics*): therefore, in this caſe the times will be very nearly equal, and conſequently the motion of the pulſes will very nearly conſpire with the motion of CD.

33. But if the ſtring CD be conſiderably different in length from AB whether greater or leſs, and alſo incommenſurable to it, the ſtrong vibrations of AB will not excite any ſound in it, becauſe the times of the vibrations of the pulſes and of CD will be ſo diſcordant, that every ſucceſſive pulſe will counteract and check the motion which the preceding pulſe had generated. Neither, when CD is greater than AB, will its ſubmultiple parts be ſufficiently agitated to produce a tone, if theſe ſubmultiples, which approach nearly to a common meaſure of the two ſtrings, be very numerous: both becauſe theſe parts are not commenſurable to AB;

and becauſe, by being ſo numerous, they will receive but very few conſpiring impulſes; as will preſently be explained at large.

34. If the ſtring CD (*fig.* 13) be exactly double of AB, and AB be ſo ſtruck as to produce a forcible tone, the chord CD will reſolve itſelf into two equal parts; each of which will ſeparately vibrate, and produce a tone in uniſon with that of AB; the middle point E remaining very nearly at reſt.

This experiment, together with ſome of thoſe which follow, is mentioned by Wallis in his Algebra, p. 466. vol. 2. and was communicated to him by Narciſſus Marſh, archbiſhop of Caſhel, in the year 1676, as a new diſcovery made by Mr. Noble and Mr. Pigot of the Univerſity of Oxford. Long after, Monſ. Sauveur communicated it to the

Royal Academy at Paris as his own diſcovery; but upon his being informed, by ſome members preſent, that Doctor Wallis had publiſhed it before, he immediately reſigned all pretenſions to the honour of it *.

This phænomenon is to be accounted for in the following manner: the pulſes propagated from AB, which ſtrike on CD, produce a double effect, as in other caſes, *viz.* a motion of the entire ſtring, and alſo a ſeries of indentures (29); the former of theſe two effects muſt be deſtroyed, for CD, being double of AB, will perform half a vibration, while AB performs an entire vibration. Therefore when CD has moved from E to F, and returned back to E, the pulſe, which ſet E in motion, will have performed an entire vibration; and therefore will be in its ſtate of greateſt condenſation going forward, when CD is in the middle point of its regreſs; and conſequently will totally deſtroy the motion of E, the middle point of CD, the impulſes being equal and contrary: for the elaſtic force with which the ſtring returns from F to E, will be equal to that with which it ſet out from E, that is, to the force of the pulſe. And the ſame may be ſaid of all the other impulſes, as to the firſt of their two effects. Therefore the point E will very nearly continue at reſt, the motion generated in it by each pulſe being deſtroyed by that immediately ſucceeding. The pulſes which ſtrike againſt *m* and *n*, the middle points of CE and ED, produce the ſimilar and equal indentures C*d*E, E*f*D; which, becauſe their latitude is double C*m* (24), will interſect each other in E, the middle point of the ſtring; and as the motions of theſe indentures, or ſubmultiple ſtrings, are exactly alike, they will conſtantly have the ſame interſection E. But farther, ſince theſe ſubmultiple ſtrings are of the ſame length with AB, they will vibrate exactly in the ſame time with it; and conſequently every ſucceeding pulſe will encreaſe the motion of theſe parts; and as no other aliquot parts of the whole ſtring vibrate in the ſame time, there cannot be any other equally powerful ſeries to croſs and break them (30); and conſequently their motions will, at length, become ſtrong enough to affect the ear. The pulſes which ſtrike on the middle points of C*m*, *m*E, E*n*, *n*D, agitate thoſe portions; but as they vibrate in half the time of the pulſe, they will only receive an impulſe every ſecond vibration; and therefore the motions of CE, and ED will predominate. And beſides, as the lengths of the aliquot parts C*m*, *m*E, E*n*, *n*D are leſs than thoſe of

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* Sir John Hawkin's Hiſtory of Muſic, vol. 3, p. 134.

CE and ED, the ſucceſſive pulſes will generate in them a leſs degree of motion (15), and conſequently theſe portions will not produce any ſenſible effect. The ſame may, *a fortiori*, be ſaid of all other ſubmultiples of CD. Though indeed we can conceive it poſſible, that the human ear might have been endued with ſuch a degree of ſenſibility, as to be capable of hearing all theſe ſounds at once *. The remaining pulſes ſtrike on the middle points of aliquant parts of the ſtring; and therefore every ſeries of indentures produced by them will be croſſed and broken by others equally powerful, and conſequently will be extinguiſhed, according to what we have ſhewn in (30).

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* But had this been the caſe, we ſhould never have had any idea either of harmony or melody; as every ſtring contains in itſelf an infinite variety of diſcords. For ſince a ſtring and its aliquot diviſions are as the numbers 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. the number of vibrations performed by them in a given time will be as the natural numbers 1, 2, 3, 4, &c. *Smith's Harmonics, prop.* 24. *cor.* 7. And conſequently this ſeries of ſounds muſt contain all the poſſible varieties of intervals. If therefore every aliquot diviſion produced a ſenſible effect by its vibration, we ſhould hear in every muſical ſtring an infinite variety of chords, diſſonant and conſonant, in ſharp and flat keys, at the ſame time. Thus would all the charms of melody be deſtroyed: and where many muſical ſtrings were ſound-

ing together, this confuſion of conſonance and diſſonance would be ſtill farther increaſed, and therefore much more ſhould we have been deprived of the perception of harmony. We have here therefore an inſtance of the admirable ſkill with which the different parts of nature have been adapted to each other by their all-perfect Contriver; who as in other caſes he appears to have conſulted the welfare of his creatures, ſo from this inſtance we ſhould infer, that he has not been leſs attentive to their innocent gratifications. Had the human ear been endued with a leſs degree of ſenſibility than it is at preſent poſſeſſed of, it is evident that we ſhould have loſt much of the delightful effects of harmony: and had it been endued with greater, we ſhould have had no perception either of melody or harmony, as we have juſt now endeavoured to prove. It appears therefore, that the ſenſibility of the human ear has juſt attained the limit that contributes moſt to the pleaſures of hearing.

Sir John Hawkins, in his Hiſtory of Muſic, has complimented archbiſhop Marſh for his exquiſite ſolution of this phænomenon; but I apprehend, that whoever ſhall be at the pains of examining into its merits, will be perſwaded that he was induced to paſs this favourable judgment on it, merely on the authority of Doctor Plott *.

35. It may perhaps be objected to the ſolution here given, that the point E is the vertex of the indenture *d*E*f*, and that as the angle *d*E*f* is equal to C*d*E, and the tending forces acting in the direction of the ſides of theſe angles equal, the point E ought to vibrate through as great a ſpace as that through which *d* or *f* vibrates. But in anſwer to this we are to obſerve, that the convexities of the indentures lie contrary

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* See his Natural Hiſtory of Oxfordſhire, ch. 9. § 199.

ways, and therefore when *d* is going forward E is returning back; as has been explained at large in paragraph 24; and therefore when *d* receives a conſpiring impulſe from the air, E will receive the contrary; conſequently the motion produced each preceding moment by the tending forces in the directions *d*E, and *f*E will be deſtroyed the next; whereas the motions of *d* and *f* are conſtantly increaſed. The true figure, therefore, of the ſtring in vibrating, will be ſuch as is repreſented in *fig*. 14; but as the motion of E is very inconſiderable, the points *e* and E may be looked upon as coincident, and the figure of the ſtring may be repreſented by *fig*. 15. For the ſame reaſon that E is nearly quieſcent in this caſe, all the nodes in other caſes of indentures will be alſo nearly quieſcent.

36. If, converſely, the ſtring CD be forcibly ſtruck, it will excite the tone

of AB; becauſe, while CD vibrates once, AB vibrates twice, and therefore will receive a concurring impulſe at the end of every ſecond vibration.

37. Strike AB the octave to the fifth above the fundamental CD (*fig.* 16), and the baſe CD will reſolve itſelf into three equal parts, and ſound the uniſon of that octave. For AB is to CD as 1 is to 3, therefore the whole ſtring cannot vibrate; for the particles of air adjacent to the ſtring are ſuppoſed to impel it only when they are in their ſtate of greateſt condenſation, that is, in the middle of their progreſs; therefore after the firſt impulſe of theſe particles on CD, there will be another when they have performed an entire vibration, for then they will be in their ſtate of greateſt condenſation again; and in the mean time CD will have performed one-third of its entire vibration, or $\frac{4}{3}$ YN; that is, if YN be tri-

ſected in *x* and *z*, and YO in *r* and *s*, CD will have moved from Y to N and from N to *z*, and therefore will be returning backward whilſt the pulſe is going forward, and therefore its motion will be checked; when it receives the next impulſe it will have got to *s* in its return, and therefore will be checked again; and at length the next impulſe it receives will be when it has returned to Y. Since therefore during the entire vibration its motion is twice checked, and only once favoured by the pulſe, it is obvious, that the whole ſtring cannot be ſufficiently agitated to produce a tone. It remains only therefore for us to conſider the motion of the indentures produced by each pulſe which ſtrikes on CD.

The three equal pulſes which ſtrike on *m*, Y, and *n*, the middle points of the portions CE, EF, FD, will produce three equal interſecting indentures;

and as their motions are exactly alike, their nodes will conſtantly continue in E and F. But farther, theſe ſubmultiple ſtrings being of the ſame length with AB, they will vibrate exactly in the ſame time with it; and conſequently every ſucceeding pulſe will increaſe the motion of theſe parts; and as no other aliquot parts of the whole ſtring vibrate in the ſame time, there cannot be any other equally powerful ſeries to croſs and break them (30); and conſequently their motion will at length become ſtrong enough to affect the ear. As to the motions produced in any other aliquot or aliquant parts of the ſtring, they cannot produce any ſenſible effect, for the reaſons aſſigned in (34).

38. If converſely the ſtring CD be forcibly ſtruck, AB will return its proper tone; becauſe it will receive a concurring impulſe at the end of every third vibration.

39. Strike forcibly AB the fifth (*fig.* 17) above the fundamental CD, and the tone generated, by the pulſes, in the quieſcent ſtring CD will be the acute octave to AB. For AB is to CD as 2 is to 3; therefore while AB performs one vibration, CD will perform two-thirds of a vibration, or $\frac{2}{3} \times 4$ YN or $\frac{8}{3}$ YN; that is, if YN be triſected in x and z, and YO in r and s; CD will deſcribe eight of theſe little parts, while AB vibrates once; therefore when CD receives its ſecond impulſe it will be in s, returning backward whilſt the pulſe is going forward; and therefore will be checked in its motion; it will receive the third impulſe when it is in z, returning alſo backward whilſt the pulſe is going forward, and therefore will be interrupted again; laſtly, it will receive a conſpiring impulſe at the end of the third vibration of AB, and the ſecond of CD. Therefore, as in

(34), the whole ſtring cannot be ſufficiently agitated to produce a tone.

It remains therefore only for us to conſider the motion of the ſeveral ſeries of indentures in the ſtring: the pulſe propagated from AB, which ſtrikes on *m* the middle point of C*p*, would excite a vibration in the ſtring C*p* equal in length to AB, which would in every vibration of AB receive a new impulſe, and conſequently would predominate, were it not that another equally forcible pulſe ſtriking on *p* the middle of *m*D, will produce an equally powerful indenture at the other extremity, which will interrupt and break the former, becauſe they are aliquant parts of the whole ſtring (30). The portions therefore of CD which will produce the predominant tone, muſt be aliquot parts of the ſtring (*ib.*) But they muſt alſo be ſuch as will receive the greateſt number of conſpiring impulſes from the aerial pulſes propagated from AB. The length therefore of theſe aliquot parts of CD muſt be a ſubmultiple of AB, that they may receive any concurring impulſe; and that they may receive the greateſt poſſible number, their length muſt be the greateſt common meaſure of the two ſtrings; that is, muſt be equal to half of AB, which produces its acute octave.

The coincidences in this experiment are only at the end of every ſecond vibration of AB; in the former, they were at the end of every ſingle vibration of the ſame ſtring: hence we ſee why, in this caſe, the tone is more languid than in the former.

It might perhaps be objected, that ſince the parts C*m*, *mp*, *p*D vibrate in half the time of AB, they ſhould deſtroy its motion; for the ſame reaſon that the vibrations of AB, in paragraph 34 (*fig.* 13), deſtroy the motion of CD, its double: but that this is contradicted by experience. The anſwer to this objection is obvious; in paragraph 34, the vibration of AB is the cauſe of the motion of CD, and the impulſes from the air on CD being equal and contrary, muſt deſtroy each others effects. But in the preſent caſe, it is the vibration of AB which generates the motion in the parts C*m*, *mp*, *p*D; and as this generated motion is very much inferior in ſtrength to that which generates it, it is evident that the pulſes propagated from theſe parts can have but a very inconſiderable effect on AB.

40. If, converſely, CD (*fig.* 17) be forcibly ſounded, AB will be reſolved into two equal parts, and will produce the ſame tone that CD produced before; that is, the acute octave of AB, or the twelfth above CD. For while CD vibrates twice, AB vibrates three

times; and therefore while CD vibrates once, AB will perform one entire vibration and alſo an half; therefore AB will receive the ſecond impulſe, from the aerial pulſes, in the middle of its regreſs, and therefore its motion will be deſtroyed; and the whole ſtring will not produce a tone. But the pulſes which ſtrike on the middle points of A*n* and *n*B agitate thoſe parts, which therefore, being to CD as 1 is to 3, will receive a conſpiring impulſe at the end of every third vibration, without any interruption from the pulſes in the intermediate time; and alſo as they are aliquot parts of the whole ſtring, there will be no ſeries of indentures to croſs and break them (30); conſequently their motion will predominate, and become powerful enough to affect the ear.

41. From theſe experiments we may derive this general concluſion, that if

two commenſurable ſtrings be of equal diameter, denſity, and tenſion, and one of them be forcibly agitated; the tone produced in the other will be that of ſuch a portion thereof as is the greateſt common meaſure of the two ſtrings. And as the vibrations of all uniſons are performed in the ſame time; the tone produced in any ſympathetic ſtring by the pulſes propagated from another, will be the ſame as in the former caſe, although the ſtrings be of different diameters, denſities or tenſions, provided they be tuned in uniſon with ſuch as agree in thoſe circumſtances.

42. Malcolm has been betrayed into a miſtake with reſpect to theſe tones, in ſuppoſing that an acuter ſtring cannot excite a tone in a graver: "In ſome caſes, ſays he, the motion of the untouched ſtring is ſo checked as never to be ſenſible, or at leaſt to give any ſound;

and in fact we know, that in no caſe is this phænomenon to be found but the uniſon, octave, and fifth; moſt ſenſibly in the firſt, and gradually leſs in the other two, which are alſo limited to this condition, that the *graver* will make the *acuter* ſound, but not contrarily." (*See his Treatiſe on Muſic, p.* 87.) Now on the contrary, we find by experiment in (39) and (40), that if the lengths of two ſtrings be to each other in the ratio of 2 to 3, in which caſe the one is a fifth to the other, the ſame tone will be excited in the quieſcent ſtring, whether it be the acuter or the graver. But for the greater ſecurity of ſucceſs, the ſtrings ſhould be tended on the ſame inſtrument.

Rouſſeau has ſpoken of theſe ſympathetic tones in his Muſical Dictionary, (*ſee article Uniſſon*); and has ſhewn how motion may be communicated from one ſtring to another which is a ſubmultiple of it: but he has not ſhewn, converſely, how motion may be communicated from the acuter to the graver; much leſs has he explained how the ſympathy is produced where the ſtrings are to each other, not as an integer to unity, but in any commenſurable ratio whatſoever.

43. The theory here delivered will enable us to decide on the origin of the minor mode, as given by Rameau. Let there be three ſtrings whoſe lengths are as the numbers 5, 3, 1, their denſity, diameter and tenſion being the ſame; if the ſhorteſt of theſe be ſounded, it will cauſe the other two to reſolve themſelves, the one into five, the other into three equal parts, each of which will tremble; and we ſhall obſerve four quieſcent nodes in the one ſtring and two in the other. This trembling, M. Rameau ſays, diſcovers a certain natural connection between

the tones produced by the whole ſtrings. He obſerves farther, that it makes no alteration in the ſubſtance of harmony, whether we uſe a certain note or any of its octaves. Take then the acute octave of the loweſt of theſe tones, and the grave octave of the higheſt, and the three numbers expreſſing the chord will be 3, $\frac{5}{2}$, 2, which is the minor mode *. But the fallaciouſneſs of this method of deriving it from the mere communication of motion in ſympathetic ſtrings, will evidently appear from what has been already demonſtrated: for the ſame manner of arguing may be applied to various other caſes, in which the reſulting chord is diſſonant. It has been ſhewn, that a ſtring, forcibly ſounded, will cauſe any other, of equal diameter, denſity, and tenſion, which is a multiple thereof, to reſolve itſelf

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* See Elemens de Muſique, p. 23. and Antonio Eximeno Dell' origine e regole della Muſica, p. 97.

into ſuch aliquot diviſions as are equal in length to the ſubmultiple ſtring. Let then the lengths of three ſtrings be as the numbers 7, 3, 1; the ſhorter of theſe being forcibly ſounded, will cauſe the other two to tremble and reſolve themſelves, the one into ſeven, the other into three equal parts. This trembling therefore, according to Rameau, intimates ſome natural connection between the tones produced by the whole ſtrings. If now we take the acute double octave of the loweſt, the three numbers expreſſing the chord will be 3, $\frac{7}{4}$, 1, which is a diſſonance. In the ſame manner, let the lengths of the three ſtrings be as the numbers 11, 3, 1; if now we take the acute double octave of the loweſt, and the grave octave of the higheſt, the reſulting chord will be denoted by the numbers 3, $\frac{11}{4}$, 2; which is alſo a diſſonance. And the like may be proved in a variety of other caſes.

SECT. III.

OF SECONDARY TONES.

44. IT was obſerved by M. Rameau about the year 1750, and afterwards by M. D'Alambert and many other muſicians, that a muſical ſtring, beſides its peculiar and fundamental tone, produces ſeveral others which are called *ſecondary*. And on this ſingle fact it was, that M. Rameau founded his ingenious ſyſtem; whence he has been ſtiled *the Newton of harmony*; as he has derived the whole from one principle, to wit, the fundamental baſe; in the ſame manner as Newton, from the ſingle principle of gravitation, has ex-

plained the moſt remarkable phænomena in phyſics *.

The fact is eaſily aſcertained, by ſtriking the baſe notes of an harpſichord or violoncello; for when the principal note has decayed and languiſhed, you will hear diſtinctly the octave above the principal, the twelfth, and ſeventeenth major. This however is not a modern diſcovery, for it was known to the writers on harmonics at leaſt ſo long ago as the year 1618. Merſennus was acquainted with theſe ſecondary tones, as may be ſeen in his chapter on the difficulties of muſic, Opera Mathem. vol. 1. p. 355: and Des Cartes, his cotemporary, in a letter to Merſennus, ſpeaks of them in the following manner: "As to the different tones which are produced at the

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* Sir John Hawkins's Hiſtory of Muſic, vol. 5. p. 385.

ſame inſtant by the ſame chord, I know no other reaſon that can be aſſigned than that which, I think, I formerly communicated to you by letter, *viz.* that while the whole ſtring performs its vibrations, its parts may have leſs ſenſible motions; which, meeting with the whole body of air already ſet in motion by the principal vibration of the entire ſtring, muſt neceſſarily render the percuſſions which it makes on the ear double, or triple, or quadruple, or quintuple, and ſo generate the octave, twelfth, fifteenth, and ſeventeenth." *Ep. P.* 2. *Ep.* 106.

M. Bernouilli accounted for theſe ſecondary tones in a manner ſomewhat ſimilar to that of Des Cartes, by ſuppoſing that every ſtring reſolves itſelf into aliquot parts, and that while the whole ſtring performs its vibrations, the ſubmultiples alſo perform their ſecondary vibrations. Thus the principal vibration produces the baſe note, that of one-half the ſtring generates the octave, that of one-third the twelfth, and that of one-fifth the major ſeventeenth. This very ingenious hypotheſis, was adopted by Euler, and after him by Antonio Eximeno, in his excellent treatiſe *Dell' origine e regole della Muſica*, printed at Rome in the year 1774. Bernouilli and Euler, on the ſuppoſition that ſtrings do reſolve themſelves in this manner into aliquot diviſions, have given us various calculations of the forms which they aſſume in their vibrations. He that wiſhes to enter into this mathematical diſcuſſion, may find it at large in the Philoſophical Tranſactions of Berlin and Peterſburgh. The grand reſult of their enquiries is, that the vibrations of a ſtring, whatever form it aſſumes, are ſynchronal; and that the time of each vibration is proportional to the length of the chord, its diameter and tenſion being given. For if the length of any uniform chord be called *a*, $\varkappa$ the weight of a portion of the ſtring whoſe length is *k*, the tending force π, and the ſpace deſcribed in a ſecond by a body falling freely from a ſtate of reſt *h*, then the time of the vibration will be $\frac{a\varkappa^{\frac{1}{2}}}{\sqrt{2hk\pi}}$, (*Acta Petrop. v.* 17. *p.* 394). Conſequently in the ſame chord, ſince all theſe are invariable quantities, the time of the vibration muſt alſo be invariable: and in different chords of the ſame diameter, denſity and tenſion, ſince $\varkappa^{\frac{1}{2}}$ and $\sqrt{2hk\pi}$ are given quantities, the times of their vibrations will be as *a*; that is, directly as their lengths.

45. But theſe ingenious writers have not favoured us with their conjectures why muſical ſtrings ſhould thus reſolve themſelves into aliquot parts; nay even their own theories do not ſeem very well to conſiſt with ſuch an idea. For

Bernouilli aſſerts, (*vide Phil. Tranſ. of Berlin. an.* 1753), " that the curve of a ſtring in vibrating will be a trochoid, or companion of a cycloid, extremely prolonged;" in which he follows Mr. Taylor, who has delivered this doctrine in his method of increments *. But there is nothing in ſuch a vibration which can poſſibly lead us to conclude, that the ſtring will reſolve itſelf into ſubmultiple trochoids. And M. D'Alembert has very properly made this objection to Bernouilli : " I demand of him firſt, ſays he, what is the reaſon that a vibrating chord ſhould be compoſed of ſeveral trochoids ? And ſecondly, which he has explained ſtill leſs, what is the cauſe that determines theſe trochoids to be conſtantly ſuch as to produce the octave, twelfth, and ſeventeenth, rather than any other ſound ?" *(See Encyclopædie, article Fondamental.)* Euler, who is more general in his ſolutions, has given us a ſcale for calculating the various figures which a ſtring will aſſume in vibrating, the figure of the ſtring before vibration being given; and particularly, if a ſtring be drawn into a triangular form ACB (*fig.* 18) before vibration, he proves, that the point C will ſo deſcend towards the baſe AB, as that the ſtring will always aſſume the figure AFGB, FG being parallel to AB. (*See Acta Petrop. an.* 1772). Hence we ſhould conclude, in a caſe of this kind, ſince it does not appear that the ſtring would be reſolved into aliquot parts, that we could never hear a ſecondary tone; which is contradicted by experiment. And the ſame objection will hold, whatever be the form of the ſtring before vibration.

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* See alſo Smith's Harmonics, ſect. 11.

46. The author of the *Elucidationes Phyſicæ* upon Des Carte's Compendium of Muſic, and after him Monſ. Mairan,

(*Sec Encycl. Yverdon*, *article Son*), has attempted to account for theſe tones on different principles. He ſuppoſes that there are in the air particles of every poſſible variety of tenſion, which by their vibrations produce all the indefinite variety of notes; that ſome of theſe will of conſequence vibrate quicker, others ſlower: and that when a ſtring is vibrating, it communicates its motion principally to thoſe particles which are of the ſame degree of tenſion with itſelf, and conſequently vibrate in the ſame time. After theſe, the particles which leaſt interrupt the chord, and which are leaſt interrupted by it, are ſuch as perform two vibrations in the ſame time that the chord performs one; becauſe their vibrations begin together after every ſecond vibration; theſe produce the acute octave. In the ſame manner, the twelfth and ſeventeenth are generated by the particles which vibrate three and five times while the chord vibrates once; each tone decaying in ſtrength according to its acuteneſs, becauſe leſs favoured by the vibrations of the chord.

But this theory is liable to every ſpecies of objection; for it not only appears to be a moſt arbitrary hypotheſis, deviſed merely with a view to ſay ſomething on this intricate ſubject, without any regard to the real conſtitution of nature; but is in fact contradicted by experience. For if theſe tones depended on the configuration or tenſion of the particles of air, our voices, in ſinging, ſhould as frequently produce them as ſtringed inſtruments. But beſides, without recurring to experience, we can from theory alſo demonſtrate that this hypotheſis is falſe; for it would follow as a direct conſequence, that admitting the air to be a fluid perfectly homogeneous, all muſical ſtrings, however different from each other in length, diameter, denſity and tenſion, would produce the very ſame tone: now on the contrary, Sir Iſaac Newton has demonſtrated, (*Prop.* 47. *L.* 2), that the pulſes vibrate according to the law of the ſounding body; and therefore that the tone will vary according to the number of vibrations performed by that body in a given time, ſuppoſing the medium homogeneous.

47. Monſ. D'Alembert has advanced another objection to this hypotheſis; and although the opinion which he controverts be falſe, as I think we have ſufficiently proved, yet it is neceſſary I ſhould endeavour to obviate this objection, as it would equally hold againſt the doctrine which I ſhall preſume to offer upon this difficult queſtion.

He ſays, that even if we ſhould admit ſuch particles to exiſt, the propoſed effect would not follow from the coincidences of the pulſes of the air with their vibrations: for as an octave ſtring vibrates twice for one vibration of the baſe, it will be as often oppoſed as favoured by the vibration of the baſe. If a ſtring be one-fifth of the principal, it will be oppoſed twice and favoured three times; thus ſuppoſe CD (*fig.* 19) one-fifth of AB, while AB vibrates once, CD will vibrate five times, therefore while the pulſe is going forward through a ſpace equal to EF, CD will move twice from G to H and from H to G; and beſides, will alſo move once from G to H; but during this time, according to D'Alembert's reaſoning, its motion while going forward from G to H will be aided by the pulſe, and oppoſed while returning from H to G; that is, it will be aided three times in running over the ſpace GH, and oppoſed twice. The ſame thing will happen during the other half of the vibration of AB; therefore during the whole vibration, it will be favoured by the pulſe while it deſcribes the ſpace GH ſix times, that is, during three vibrations; and will be oppoſed four times, that is, during two vibrations. And in general any ſubmultiple ſtring whoſe denominator is an uneven number, will, while the multiple ſtring vibrates once, be oppoſed during the leſs, and favoured during the greater of the two portions of time into which the whole time is divided, which portions are denoted by the two moſt nearly equal integers into which the whole number of vibrations can be divided. Hence, he ſays, the octave ought to be heard leſs diſtinctly than the twelfth or ſeventeenth, or rather not heard at all: contrary to experience. *See Encyclopædie.*

But Monſ. D'Alembert ſeems to me to have here fallen into an error, in his manner of conceiving the effect of the pulſes: for we are to conſider the particles of air as giving their impulſe only when they conſtitute a pulſe, that is, when they are in their ſtate of greateſt condenſation, or in the middle of their progreſs; according to our reaſoning in (12) *. Now in the caſe of the octave, the firſt impulſe made on AB (*fig.* 13), by the pulſes propagated from CD, will be when the adjacent particle is in the midſt of its progreſs; and it will not receive another till that particle has got into the ſame poſition again: and as AB performs exactly two vibrations while CD performs one, it is obvious, that when the particle gets to the middle of its progreſs, AB will be in the inſtant before the middle of its progreſs; and conſequently will receive a conſpiring impulſe as before. Therefore AB, inſtead of being as often oppoſed as favoured by the pulſe, will not be oppoſed at all. And the ſame mode of reaſoning may be applied in the caſe of any other ſubmultiple ſtring.

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* See alſo Smith's Harmonics, ſect. 1. art. 12.

48. The authors of the *Encyclopædia Britannica* have advanced a new ſolution of theſe phænomena: "With regard to the production of the different tones from the baſs ſtring of an harpſichord, ſay they, we can only offer a conjecture, which is, that as the ſtrings of muſical inſtruments are faſtened at both ends and very tenſe, the vibrations of the middle parts muſt be performed much more eaſily than thoſe towards the extremities; conſequently, as vibrations muſt have a certain degree of ſtrength before a ſound is produced, the middle parts of the ſtring may vibrate ſo as to produce a ſound, while the extremities have loſt that power. This will be equivalent to ſhortening the ſtring, and conſequently the tone muſt grow gradually more acute." (*See*

article Acouſticks). Now were this hypotheſis true, according as the motion of the ſtring languiſhes, and its extreme portions become gradually quieſcent, we ſhould hear a continued ſucceſſion of all the poſſible tones which are more acute than the original tone of the entire ſtring; directly contrary to experience.

49. Since therefore the theories, hitherto advanced reſpecting theſe tones, appear to be unſatisfactory and imperfect, we muſt have recourſe to ſome other principle for their ſolution. And perhaps none will appear more probable than that which has been already applied to ſympathetic tones: for the ſimplicity obſervable in the works of nature, gives conſiderable forcc to an hypotheſis which can be applied with equal facility to the ſolution of many different phænomena.

When a muſical ſtring performs its vibrations, it excites pulſes in the air which are propagated in all directions; theſe ſtriking againſt all the ſurrounding bodies, every point of impulſe becomes a new centre, from whence pulſes are alſo propagated in every direction; ſome of which will conſequently be propagated towards the ſounding body, and will ſtrike along its entire length, in the ſame manner as if they had been propagated from another ſtring in uniſon with the propoſed one. Thus will the preſent caſe become exactly of the ſame nature with that mentioned in (30). The pulſes which ſtrike againſt the middle point of the whole ſtring, and which at the time of their impulſe on the ſtring are moving in a contrary direction to it, will only diminiſh the quantity of motion of the entire ſtring: thoſe which ſtrike againſt the middle points of the halves, will endeavour to agitate theſe halves, the motion being communicated equally on each ſide of the point of impulſe (24); and therefore will not be interrupted, becauſe they are aliquot parts of the ſtring (30). Now as the time of the vibration of the halves is half the time of that of the entire ſtring, the pulſe generated by the whole ſtring will coincide with every ſecond vibration of the halves, and therefore will not interrupt them, and conſequently will, at length, increaſe their motion ſo much as to enable them to produce a tone. Thoſe particles which ſtrike againſt ſuch points of the ſtring as are not the middle of aliquot parts, will excite vibrations that will both be interrupted by other ſeries equally powerful, and by the original pulſe of the whole ſtring, and therefore will produce no ſound (*Ib.*) Thoſe pulſes that ſtrike againſt the middle points of each of the third parts of the ſtring will agitate theſe parts, which will receive a concurring impulſe at the end of every third vibration, and therefore their vibrations may at length become ſtrong enough to affect the ear. And the like holds of other aliquot parts ; but the ſhorter the portion, the fewer will be the coincidences of the pulſes with the vibration of that part, and conſequently the weaker the tone. The greater the tenſion of the fundamental the more difficult it is to hear the ſecondary tones, and the difficulty increaſes in the direct ratio of the number of vibrations performed in a given time by the fundamental. For to produce tones of equal ſtrength in ſtrings of the ſame length, diameter, and denſity, the impulſe muſt be in the ſubduplicate ratio of the tending force (15) ; that is, reciprocally as the time of the vibration, or directly as the number of vibrations performed in a given time. And if the ſtrings be of equal diameter, denſity, and tenſion, but their lengths different, a given number of equal impulſes will produce a ſtronger tone in that which is the longer. (*Ib.*)

The phænomena therefore of ſecondary tones appear to be preciſely the ſame with thoſe of ſympathetic tones; and are to be accounted for on the very ſame principle: they are always the octave, twelfth, fifteenth, and ſeventeenth major; becauſe theſe notes are produced by the half, third, fourth, and fifth parts of the entire ſtring; they gradually decay in ſtrength, according as the tone is ſharper, becauſe the number of concurring impulſes diminiſhes, as the number of the aliquot diviſions increaſes; and we hear them moſt diſtinctly in the lower ſtrings of the violoncello or harpſichord, becauſe the length is greater or the tenſion leſs in theſe ſtrings than in the higher.

50. Beſides thoſe tones mentioned above, M. de Betizhy, on the teſtimony of Rameau, affirms that three others are alſo perceived; namely, the octave of the twelfth, a ſound which he called *the loſt ſound*, and the triple octave of the fundamental, (*ſee Expoſition de la Theorie & de la Pratique de la Muſique, Part.* 2. *Cap.* 2. *Art.* 1.) Now the firſt of theſe was produced by one-ſixth, the ſecond by one-ſeventh, and the third by one-eighth part of the entire ſtring.

51. As the theory of ſympathetic tones (43) enabled us to diſcover Rameau's error in deriving the minor mode from the mere communication of motion to quieſcent ſtrings; ſo the phænomena of ſecondary tones will ſhew us, that D'Alembert has fallen into a ſimilar miſtake, in his account of the origin of both modes, major and minor; which he has endeavoured to

derive from theſe ſecondary tones in the following manner:

The fundamental tone *ut* of any ſonorous body being ſounded, we find it always accompanied by its twelfth, and ſeventeenth major, that is, by the octave of *ſol*, and the double octave of *mi*: the moſt perfect chord therefore which can be formed with any note is that which conſiſts of theſe ſecondary tones, denoted by the numbers $\frac{1}{5}$, $\frac{2}{3}$, ſince this chord is the work and choice of nature. Now as it makes no alteration in the ſubſtance of harmony whether we uſe a certain note or any of its octaves, we may ſubſtitute the grave octave of the twelfth, and grave double octave of the ſeventeenth inſtead of theſe tones themſelves: the reſulting chord therefore *ut*, *mi*, *ſol*, will be denoted by the numbers 1, $\frac{4}{5}$, $\frac{2}{3}$, which is the major mode.

This argument is founded on the hypotheſis, that thoſe tones which are naturally connected, form the moſt pleaſing conſonance. But the fallacy of this principle is evident from the teſtimony of M. Betizhy, who informs us, as we have already obſerved, that a muſical ſtring produces, together with the twelfth and ſeventeenth, a tone which he calls *the loſt ſound*, becauſe it deſtroys the harmony. It appears therefore, that the natural connection of the ſecondary tones, is no proof that they will form the moſt pleaſing conſonance; but that it is a mere mechanical effect, to be accounted for on phyſical principles, as we have attempted to do in the former part of this ſection.

The minor mode he derives from the major, in the following manner: in the modulation *ut*, *mi*, *ſol*, which is the major mode, denoted by the numbers 1, $\frac{4}{5}$, $\frac{2}{3}$, or the equivalent numbers as to harmony, 1, $\frac{1}{5}$, $\frac{1}{3}$, the ſounds *mi* and *ſol* are ſo proportioned one to the other, that the principal ſound *ut* cauſes both of them to reſound; but the ſecond ſound *mi* does not cauſe *ſol* to reſound, which only forms the interval of a minor third. Let us then imagine, that inſtead of the ſound *mi*, another ſound is ſubſtituted between the ſounds *ut*, *ſol*, which, as well as the ſound *ut*, has the power of cauſing *ſol* to reſound, and which is, however, different from *ut*. Now a ſtring which is quintuple of that which produces *ſol*, will have this effect; if then we ſubſtitute the tone of this ſtring inſtead of *mi*, the reſulting chord will be denoted by the numbers 1, $\frac{5}{3}$, $\frac{1}{3}$, or by the equivalent numbers as to harmony, 1, $\frac{5}{6}$, $\frac{2}{3}$; which is the minor mode.

The harmony therefore of the minor mode, according to D'Alembert, ariſes from this, that both the firſt and ſecond tones of the chord cauſe the other to reſound. But that this is an inſufficient account of the matter will appear from obſerving, that the ſame effect may take place in other chords which are diſſonant. We have ſeen before, that any ſtring, forcibly ſounded, generates beſides the twelfth and ſeventeenth major, a ſound which is called *the loſt ſound*, produced by the ſeventh part of the entire ſtring. If therefore we take a ſtring ſevenfold of that which produces *ſol*, it will cauſe *ſol* to reſound; ſubſtitute therefore the tone of this ſtring inſtead of *mi*, and the three numbers denoting the chord will be 1, $\frac{7}{3}$, $\frac{1}{3}$, in which both the tones denoted by 1 and $\frac{7}{3}$ cauſe the other to reſound; ſubſtitute now for the intermediate tone its double octave, and the reſulting chord will be denoted by the numbers 1, $\frac{7}{12}$, $\frac{1}{3}$, which is a diſcord, although poſſeſſed of thoſe qualities which render the minor mode harmonious. In the ſame manner, a ſtring elevenfold of $\frac{1}{3}$ would cauſe it to reſound, if our organs of hearing were a little more ſenſible than they are, conſequently the chord 1, $\frac{11}{3}$, $\frac{1}{3}$; or, which is the ſame thing, the chord 1, $\frac{11}{12}$, $\frac{2}{3}$ is poſſeſſed of thoſe qualities which render the minor mode harmonious, and yet is diſcordant. The origin therefore of the modes, according to D'Alembert, however ingenious, is certainly fallacious, as we have now proved; and is farther made evident by Antonio Eximeno, in his admirable treatiſe on Muſic, where he has ſhewn, that a cloſe adherence to the principles of the French muſicians is attended with infinite violations of harmony; and that theſe ingenious writers could not have given us ſuch excellent rules for practice, if theſe rules had not deſtroyed their theory *.

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* Ant. Eximeno, Lib. 1. Cap. 5. § 5.

52. D'Alembert has objected, that the ſecondary tones cannot ariſe from any ſubmultiple diviſions of the ſtring, becauſe the extremities of theſe parts are moveable; which is not the caſe of muſical ſtrings: "We might eaſily conceive, ſays he, how a chord ſhould produce, beſides the principal ſound, the octave, twelfth, and ſeventeenth, if the points of the chord which form the extremities of the aliquot parts were really nodes or fixed points, ſo as that the parts of the chord compriſed between the nodes, ſhould perform in the ſame time, the firſt, one vibration; the ſecond, two; the third, three; the fourth, four; the fifth, five, &c. In this caſe we might conſider the ſtring as compoſed of five different parts, placed in a right line, each immoveable at both its extremities, and forming by their different lengths this ſeries or progreſſion, 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c. But experience demonſtrates that this is not the caſe. In

a chord which performs its vibrations freely, we obſerve no other nodes, or points abſolutely at reſt, except the extremities." *Encyclopedie, article Fondamental.*

But it is not neceſſary that a muſical ſtring ſhould have its extremities fixed and immoveable, as will appear from the following experiments: To a hook A (*fig.* 20) was faſtened a wire ſtring AB, and from the extremity B was ſuſpended a weight ſufficiently heavy to render it muſical; I then faſtened the point B, but in ſuch a manner as that the weight ſhould be totally exerted in tending AB; I then ſtruck the ſtring and obſerved the tone: then letting the weight hang freely, I ſtruck it again, and though the weight vibrated without any interruption, yet ſtill the note continued the ſame as before. To prove the ſame thing when both ends were in motion, I ſcrewed on a plate of metal at C, which ſtopping the vibration there, the muſical ſtring was reduced to the length CB; which produced the ſame note whether both the extremities C and B were faſtened, or permitted to vibrate freely, by the impulſe made on CB. Hence we ſee, that whether one, or both ends of a muſical ſtring be moveable, it will ſtill produce the ſame tone as when they are both abſolutely at reſt, provided its length and tenſion continue the ſame.

53. But M. D'Alembert proceeds to object: "It is true, ſays he, that conſidering the nodes as moveable, and alſo the vibrating chord as compoſed of ſeveral trochoids, the different points of this chord perform their vibrations in different times. But it is eaſy to ſee, that this difference of vibrations cannot ſerve to explain the multiplicity of ſounds. In fact, let us ſuppoſe, for the greater ſimplicity, and the eaſier

conception of this matter, that the chord forms only two equal trochoids, ſo as that the middle point of the chord ſhall be the common extremity of the two trochoids; we admit that whilſt this middle point of the chord performs one vibration, the middle point of each of the ſubmultiple trochoids will perform two: but it is eaſy to ſee, that each of theſe vibrations will not be performed in an equal time, and therefore that the reunion of the two vibrations cannot produce the octave of the principal ſound, which is given by the middle point of the chord. We conclude therefore, that the different vibrations of the different points of the ſtring are inſufficient for the explication of the variety of ſounds which it produces. This is not all: if the middle point of the chord performs one vibration, whilſt the middle points of each of the trochoids perform two, it is eaſy to ſee, that the other points will partake more or leſs of the law of the motion of theſe two, according as they are nearer to, or farther from them. Thus, properly ſpeaking, the law of the vibrations of each point will be different, and each ſhould produce a particular tone, which, by its mixture with the others, muſt generate a confuſed harmony, and kind of cacophony. Why then, ſays he, does not this happen? And why does the ear diſtinguiſh thoſe tones only, in the ſound of the ſtring, which form a perfect concord?" The former part of this objection has been obviated by Bernouilli, in the Berlin Tranſactions for the year 1755, and he has there ſhewn, that D'Alembert has fallen into an error, in not referring the time of the vibration of the ſubordinate ſtrings to their moveable axis. Euler has alſo demonſtrated, as we have already obſerved (44), that the vibrations of a ſtring, whatever form it aſſumes, are iſochronal; conſequently, that whatever be the form of the ſubordinate trochoids, each of their vibrations muſt be performed in an equal time. But without having recourſe to this abſtract reaſoning, we can evidently ſee, from the laſt mentioned experiment, the inconcluſiveneſs of this objection: for it is there proved, that the tone of a ſtring is not affected by the motion of its extremities; and therefore if the whole ſtring be reſolved into two equal parts, they muſt generate the acute octave of the entire ſtring, whether it performs its own ſlower vibrations at the ſame time or not.

The latter part of the objection, which denies the poſſibility of a point's performing different ſpecies of independent vibrations at the ſame time, is overturned by our obſervation, that the human ear is capable of accurately diſtinguiſhing many different ſounds which ſtrike it at the ſame inſtant; whence it follows, that the fibres, by which theſe ſenſations are conveyed, muſt be capable of many different vibrations at the ſame time; otherwiſe a multiplicity of concurring ſounds would excite but one confuſed ſenſation, reſulting from their compounded effect. And that ſeveral different ſpecies of vibrations may actually exiſt in the ſame body at the ſame time, undiſturbed and abſolutely independent of each other, may thus be demonſtrated. But obſerve that I ſpeak here only of the acceleration of points, according to Sect. 2. L. 2. Prin. not of the mutual attraction of bodies.

Let the point D (*fig.* 28) be attracted towards the point T with a force varying in the direct ratio of the diſtances DT, and D will deſcribe an ellipſis whoſe centre ſhall be in T, (*cor.* 1. *prop.* 10. *lib.* 1. *Prin.*) Let now a third point S attract the two points D and T, with accelerating forces, varying according to the ſame law. The force SD, (by *cor.* 2. *Laws of Motion*), is reſolved into the forces DT, ST. The force DT by which D is urged towards T, being proportional to the diſtance, the point D will ſtill continue to deſcribe an ellipſis whoſe centre ſhall be in T, as before, but with a quicker motion. (*See prop.* 64. *lib.* 1. *Prin.*) The remaining accelerating force ST acting on D, and the accelerating force ST which acts on the point T, attracting thoſe points equally, and in directions parallel to ST, do not at all change their ſituations with reſpect to each other, but cauſe them to approach equally to a line which we may imagine drawn through the point S, and perpendicular to the line TS. But that approach will be hindered by giving the whole plane in which D performs its motion a projectile impulſe, in which caſe the motion of D with reſpect to T will not be diſturbed, (*cor.* 5. *Laws of Motion*), and the point T will deſcribe an ellipſis whoſe centre ſhall be in S, (*cor.* 1. *prop.* 10. *Prin.*) Let now a fourth point V attract the points S, T, D with forces in like manner, directly proportional to the diſtance, and by a like argument it will be demonſtrated, that the point S will deſcribe an ellipſis whoſe centre ſhall be in V; the motions of D and T remaining the ſame as before. And by the ſame method more ellipſes may be added at pleaſure: thus will the point D perform ſeveral different periodical revolutions at the ſame time, the firſt with reſpect to T as its centre, the ſecond with reſpect to S, the third with with reſpect to V, and ſo on. Let now the minor axes of all theſe ellipſes vaniſh, the major axes remaining unvaried, and the different periodical revolutions will be converted into as many different vibrations, which will be performed according to the law of a cycloidal pendulum, (*prop.* 38. *lib.* 1. *Prin.*) that is, according to the law of a ſounding body (12). It appears therefore, that a point of a muſical ſtring may perform ſeveral different and independent vibrations at the ſame time; which was the thing to be demonſtrated.

M. Bernouilli has given us the following ſimple demonſtration of the ſame truth in the following manner: Let the body A (*fig.* 29) be attracted towards the point B with a force directly as the diſtance, this body will perform its iſochronal vibrations with reſpect to that point, (*prop.* 38. *lib.* 1. *Prin. Math.*) Let us now ſuppoſe that, whilſt A vibrates in this manner, the entire line in which it performs theſe vibrations is equally accelerated towards the point C, ſo as that the ſyſtem

AB may ſuffer a common gravitation towards C, which may alſo be directly proportional to the diſtance BC; this gravitation will not diſturb the motion of A with reſpect to B, (*cor.* 5. *Laws of Motion*), and the ſyſtem will perform its vibrations in reſpect to C, according to the ſame law with which A vibrates in reſpect to B. We next may ſuppoſe that the three points A, B, C ſuffer in each inſtant an equal gravitation towards a fourth point D, proportional to the diſtance CD. In this manner we may multiply indefinitely the gravitations of the body A towards different centres, and it will thus perform an indefinite number of independent vibrations in reſpect to theſe different centres. *See Berlin Tranſactions, An.* 1755.

54. In theſe ſecondary tones we ſometimes hear the twelfth and ſeventeenth, when we cannot hear the oc-

tave, though from our reaſoning on this ſubject we might naturally conclude, that the tone of half the ſtring ſhould be heard more diſtinctly than any other ſecondary tone, becauſe half the ſtring receives a greater number of conſpiring impulſes from the air in a given time than any other of its aliquot parts. But in theſe feeble notes we cannot always diſcern the octave, becauſe it produces ſo ſmooth a coincidence with the baſe note as not to be diſtinguiſhable from it. Thus we find, that in an harpſichord or organ accurately tuned, the various parts of the diapaſon or octave ſtop are not diſtinctly perceived.

55. We have hitherto explained theſe phænomena of muſical ſtrings from the principle of the aerial pulſes, and ſhewn that the reaſon of their reſolving themſelves into aliquot parts is, that theſe parts vibrate in ſuch times, as to

receive the greateſt poſſible number of ſucceſſive conſpiring impulſes from the air, and at the ſame time not to interfere with, and diſturb, each others motions. Now to ſhew that this is the true cauſe, and not any property in a ſtring of reſolving itſelf into *equal* parts, I procured a wire ſtring AB, (*fig.* 21), part of which, CB, I got drawn down to a conſiderably leſs diameter than that of the part AC: this compound ſtring was faſtened at A, and tended by a ſcrew at B. By a moveable bridge I ſo divided the ſtring at D, as that the unequal portions AD, DB were in uniſon. I then removed the bridge, and ſounded ſtrongly, on a violin, the note which each of theſe parts, ſeparately, would produce; and the compound ſtring returned the uniſon. I then placed a light ſlip of paper at D, another on the middle point of the portion AD, and a third on the middle of DB; and on ſounding the ſame note as before, the two latter papers fell off, and that at D remained on the ſtring; which ſhewed that the ſtring had reſolved itſelf into two unequal, but uniſonal portions, AD, DB. But independently of this latter experiment, I might have been certain, that the point D was comparatively at reſt, for no other diviſion of the ſtring could have produced the note which was ſounded on the violin. Hence then we find a ſtring reſolve itſelf into two unequal portions; conſequently there is no property in ſtrings of reſolving themſelves into aliquot parts only, any otherwiſe than as thoſe parts produce the ſame tone, and conſequently vibrate in the ſame time, ſo as to receive the greateſt poſſible number of conſpiring impulſes from the pulſes of the air.

SECT. IV.

OF ACUTE HARMONIC TONES.

56. THE phænomena of acute harmonic tones were, as far as I can learn, firſt exhibited to the public by M. Sauveur, who performed many experiments relative to them, before the Royal Academy of Sciences in the year 1701.

57. If you preſs your finger lightly at D, (*fig.* 22) on the ſtring of a violin or violoncello, ſo as to divide it into two parts, which may be to each other as 1 to 2, and draw the bow gently over the ſtring between D and B, you will

hear what is called an *harmonic* tone, which will be in uniſon with the note produced by one-third of the whole ſtring; that is, it will be the octave to the fifth above the fundamental. M. Sauveur made this experiment.

The cauſe of the effect is this: the impulſe made on *o*, the middle of DB, endeavours to produce not only a vibration of the whole ſtring, but alſo an *undulation*, becauſe the impulſe is made only on *one point*, (25). The finger preſſing on D prevents the vibration of the entire ſtring; there remains therefore the undulatory motion only, which is propagated under the finger at D to the adjacent portion of the ſtring. And as in this caſe the effect is merely an undulation, the motion being communicated from one extremity of the ſtring to the other, the firſt wave will endeavour to generate a ſeries of equal waves, whoſe convexities ſhall lie alternately on oppoſite ſides of the axis (26). Now DB muſt be the firſt wave, becauſe the impulſe is made on *o* its middle point, and the extremity B is immoveable; therefore as DB makes an effort to excite an undulation in AD, it will effect it by repeated eſſays, if AD be capable of ſuch a motion. But AD, being double of DB, is ſuſceptible of ſuch a motion; conſequently this undulatory motion will be regulary propagated from DB to DC, and from DC to DA, each wave vibrating in the ſame time, and the convexities lying ſucceſſively on oppoſite ſides of the axis. And becauſe each vibrating part is one-third of the whole ſtring, the note will be the twelfth major above the fundamental. If the bow be drawn gently along at *m* or *n*, the effect will be the ſame; becauſe DB will be neceſſarily ſet in motion, and therefore communicate its motion as before.

58. To prove that the motion by which harmonic tones are produced, is merely undulatory, draw the bow lightly along the middle of *o*B, and the harmonic note will be the octave of the former; that is, the nineteenth above the baſe note. For the breadth of the firſt wave is *o*B (27), and it will endeavour to generate a ſeries of *equal* waves, whoſe convexities ſhall lie ſucceſſively on oppoſite ſides of the axis (26). Now the ſtring is ſuſceptible of ſuch motions, becauſe *o*B is an aliquot part of the whole ſtring (27), *viz.* one-ſixth, and the obſtacle, preſſing at D, lies at the interſection of two of the waves, and therefore does not interrupt the ſeries. Conſequently the harmonic note, being produced by one-ſixth of the whole ſtring, will be the octave of the former note, that is, it will be the nineteenth above the fundamental. We may obſerve in general that ſlight preſſures on any or all the nodes, or interſections

of the waves, cannot diſturb the undulation, for the reaſon juſt now aſſigned.

This experiment, where there is but one ſlight preſſure at D, and the bow drawn along the middle of *o*B to produce the nineteenth, requires that the ſtring ſhould be of a conſiderable length; otherwiſe the entire portion DB will be ſet in motion, and produce its proper tone.

59. A beautiful experiment mentioned by Bernouilli will confirm this reaſoning. He contrived a wheel with teeth which ſtruck againſt a muſical ſtring, ſo as that it ſhould receive two impulſes in the time of one vibration of the entire ſtring. The effect of theſe repeated ſtrokes was, that the ſtring viſibly reſolved itſelf into two equal convexities lying on oppoſite ſides of the axis. For the impulſes endeavour-

ed to generate a wave which ſhould vibrate in half the time of the vibration of the whole ſtring; and this wave endeavoured to produce another contiguous to it whoſe convexity ſhould always lie on the oppoſite ſide of the axis (26). Therefore as the ſtring was ſuſceptible of ſuch a motion, it was at length impreſſed by the repeated efforts of the wheel.

But without any preſſure whatſoever being made on any part of the ſtring, we can often, on long ſtrings, produce theſe acute harmonics. For according to the force with which the bow is moved, and the point where it is drawn along, the firſt waves excited by the impulſes will be of different latitudes, which producing their proper ſeries of equal waves will generate different harmonic notes. Thus by drawing a bow very lightly and rapidly towards the end of the ſtring of a violoncello, we frequently produce the octave, twelfth, fifteenth and ſeventeenth. But any of theſe harmonics will be more certainly generated in this manner, without any ſlight preſſure on the ſtring, if the preſſure be firſt made, and the harmonic note produced in the uſual way; and if, immediatly after, the bow be lightly drawn along, near the middle of the extreme aliquot parts; for the motion, generated by means of the preſſure, ſtill ſubſiſting in the ſtring in ſome ſmall degree, will be eaſily renewed by the impulſe on the middle of either of the extreme waves. The motion in theſe caſes is evidently undulatory, as it is communicated from one extremity of the ſtring to the other.

60. Diderot has accounted for this phænomenon in another manner, (*ſee his Oeuvres Philoſophiques*, *tom* 6, *p*. 50); he conjectures, that the firſt vibrations impreſſed on a ſtring by an impulſe,

are more rapid the more diſtant the point of impulſe is from the middle; and that it is only after a certain number of vibrations that the ſtring aſſumes ſuch a form, as that all its points will arrive at the axis at the ſame inſtant; in which caſe alone, he ſays, the vibrations become iſochronal. But Euler, (*Acta Petrop. V.* 17, *p.* 394) has demonſtrated, that the time of the vibration is the ſame in all caſes, and that it by no means depends on the figure which it aſſumes in vibrating. Beſides it is not only the *part* of the ſtring, but the *manner* in which it is ſtruck, that renders the tone more acute. And were Diderot's hypotheſis true, we ſhould as frequently hear diſcords as concords; for certainly there is no reaſon why, in theſe accidental variations of the velocity, thoſe degrees only ſhould be produced on which concords depend. But we find that no other notes are ever produced in this manner, but ſuch as appertain to aliquot diviſions of the ſtring.

Some authors maintain, that the only reaſon which can be aſſigned for the ſame ſtring returning theſe different tones, muſt be the different force of its ſtrokes upon the air. In one caſe, ſay they, it has double the tone of the other; becauſe, upon the ſoft touches of the bow, only half its elaſticity is ſet in motion. (*See Encyclopæd. Britan. article Acouſticks.*) What is meant by ſetting in motion only half, or one-third, or any other fraction of the elaſticity of the ſtring, I confeſs is unintelligible to me. But certainly harmonic tones may be produced in this manner without preſſure, which are much ſtronger than the dying tones of the entire ſtring. And farther, according to this theory, as on Diderot's, we ſhould hear all manner of diſcords as frequently as the octave, twelfth, or ſeventeenth; for there is no reaſon why theſe accidental touches of the bow ſhould not happen to ſet in motion any other fraction of the ſtring's elaſticity, whatever may be meant by that expreſſion, as frequently as its aliquot diviſions.

61. That the reſolution of ſtrings in harmonic tones into equal parts, ariſes from an undulatory motion, in which the waves muſt conſequently vibrate in the ſame time, and their convexities lie ſucceſſively on oppoſite ſides of the axis (26), will farther appear from this experiment; that if you cauſe the parts to vibrate in different times, and conſequently deſtroy the undulation, there will be no harmonic tone generated. Thus if the bow be drawn along the node C (*fig.* 22), no harmonic note will be produced; for the bow then agitates the portion AD, which endeavours to communicate a ſimilar motion

to DB, ſo as that the waves ſhould ſucceſſively lie on oppoſite ſides of the axis; but as DB vibrates in half the time of AD, ſuch a motion cannot be impreſſed, and therefore there is no tone. Again, DB, vibrating by the motion communicated to it under the ſlight preſſure at D, endeavours to propagate a ſimilar ſeries of waves to AD, but as all AD is kept in motion by the friction at C, neither can this ſubordinate motion be impreſſed on it. Therefore no regular motion can be generated, the two parts on either ſide of the preſſure counteracting each others effects: conſequently there can be no harmonic tone.

62. If the bow be drawn very violently along *m* or *n*, the middle points of AC and CD, even in this caſe there will be no harmonic, for the ſame reaſon as before; becauſe the friction is ſo violent as to agitate the whole ſtring

AD, and not to admit of its reſolution into waves; therefore there will be no muſical tone, as in the caſe where the friction was at C.

63. But if the bow be drawn gently along the middle of any of the aliquot parts, ſuppoſe at *m* or *n*, ſo as not to agitate violently the entire portion AD, but to admit of the ſtring's reſolution into theſe aliquot diviſions, there will be the ſame harmonic tone produced in all the different caſes, the preſſure continuing on the ſame point C.

64. If AD be not a multiple of DB (*fig.* 31), but yet commenſurate to it, as for inſtance, if AD be to DB as 3 to 2, the harmonic tone will be in uniſon with that portion of the ſtring which is the greateſt common meaſure of the two parts on either ſide of the preſſure; that is, in this caſe it will be the octave above DB, or

the ſeventeenth above the fundamental. M. Sauveur made this experiment. The cauſe of the effect is this: the friction being made nearly on the middle of Bm, will endeavout to agitate the whole ſtring, and alſo to generate a ſeries of equal waves, whoſe latitude ſhall be equal to Bm (27): the whole ſtring cannot be ſet in motion, becauſe of the preſſure at D, there remains therefore the undulation only; and this ſeries of waves will not be interrupted by the light preſſure at D, becauſe it is the extremity of one of the waves. Thus will the whole ſtring be reſolved into five parts, each cauſing the convexity of the contiguous wave to lie on the oppoſite ſide of the axis; and as theſe parts are equal, and therefore their vibrations ſynchronal, they will not interrupt each other's motions, and conſequently this undulation will be continued undiſturbed; therefore the harmonic will be in uniſon with the tone of one of theſe five equal parts, or the octave to DB; that is, the ſeventeenth above the fundamental.

Even if the entire portion DB were at firſt ſet in motion, but that agitation not violent, yet the motion muſt terminate in ſuch an undulation as we have deſcribed: for DB being ſet in motion, would generate another wave D*o* equal and contiguous to it; but D*o* could produce no other equal wave; A*o* would then be agitated, and would endeavour to generate a ſeries of recurring waves (27), whoſe common latitude would be A*o*; and this latitude being an aliquot part of the whole, this motion would take place in the ſtring.

65. If the bow be drawn along the node *m*, there will be no harmonic; for the reaſon mentioned in (61).

66. The ſame harmonic tone may be produced by preſſing the ſtring in ſeveral different parts; provided the greateſt common meaſure of the two parts of the ſtring on either ſide of the preſſure be the ſame in theſe different caſes (64). Thus in the preceding experiment the harmonic will be the ſame whether the preſſure be at *m*, D, *n*, or *o*.

67. That the cauſe of harmonic tones is the ſynchroniſm in the vibration of the portions of the ſtring, by which means the undulation is continued undiſturbed, the convexities of the waves lying ſucceſſively on contrary ſides of the axis; and that they do not depend on any property in the ſtring of reſolving itſelf into equal parts, will appear from the following experiment: Let the ſtring AB (*fig.* 23) be not of the ſame diameter throughout, but the part AD ſomewhat thicker than DB; and let the light preſſure be near D, ſup-

poſe at *d*, ſo as that the tone of *d*B ſhall be the octave above that of A*d*; the ſtring will in this caſe be reſolved into three *unequal* uniſonal parts. For the wave *d*B endeavours to produce another in the contiguous portion *d*C, whoſe convexity ſhall always lie on the oppoſite ſide of the axis; and this portion *d*C will endeavour to effect the like in AC. Therefore if the ſtring be capable of having ſuch a motion excited in it, it will be effected by the repeated efforts of *d*B; but as *d*B is the acute octave to A*d*, A*d* may be reſolved into two uniſonal parts with *d*B; therefore the three parts AC, C*d*, *d*B will vibrate in the ſame time; and the ſtring A*d* being ſuſceptible of the motion which *d*B endeavours to excite in it, that motion will in fact be excited.

68. The ſame mode of reaſoning may be extended to thoſe caſes, where the ratio of the number of vibrations per-

formed by one part of the unequally thick ſtring, to that of the number of vibrations performed in the ſame time by the other, is not as unity to any whole number, but in any commenſurable ratio; and the tone produced will be in uniſon with that of a ſtring which is the greateſt common meaſure of two ſtrings of the ſame diameter, denſity and tenſion, which are in uniſon with the two parts into which the irregular ſtring is divided.

To confirm this by experiment; I took the compound ſtring mentioned above, and by a moveable bridge divided it ſo in F (*fig.* 24), that the portion AF ſounded a fifth above the tone of FB; then removing the bridge, and gently preſſing the ſtring at F, and drawing a bow lightly along it, an harmonic tone was produced which was the acute octave to that fifth. Here the portion AF was reſolved into two equal parts; and the portion FB into three uniſonal parts, which muſt have been unequal, becauſe the ſtring FB is compoſed of two cylinders of different diameters.

69. To carry this experiment farther; I cauſed the thinner ſtring CB (*fig.* 24) to ſound exactly a major third above AC; and lightly preſſing the ſtring at C, and drawing the bow gently over it, I heard the acute double octave to the tone of CB. Here the ſtring CB reſolved itſelf into four uniſonal parts, and AC into five; for the major third is to its fundamental as 4 to 5, the greateſt common meaſure of which is unity; and one-fourth uniſonal part of CB ſounds the double octave above it. But the parts of AC muſt have been ſhorter than thoſe of CB, becauſe of the greater diameter of AC.

70. Again, inſtead of placing the bridge at C, I placed it a little nearer to B, as at G, and fixed another bridge firmly at M, ſo that GB ſounded an acute octave to the major third above MG; then removing the bridge at G, and preſſing the ſtring there lightly, I drew the bow gently over it, and heard the double octave to the major third above MG. Becauſe the acute octave to the major third above MG is to the tone of MG as 2 to 5, the greateſt common meaſure of which is unity; but one-half of GB is the acute octave to GB, and conſequently the double octave to the major third above MG. Here GB was reſolved into two *equal* parts, and MG into five *uniſonal* and conſequently *unequal* parts as before.

From this variety of experiments we ſee, that harmonic tones are produced by an undulation; that the tone produced is to the fundamental as the greateſt common meaſure of the numbers expreſſing the proportion of the number of vibrations performed in a given time by the two portions of the ſtring on either ſide of the preſſure, is to their ſum; and that they can only take place when the vibrations of the waves are ſynchronal. We may obſerve, that the acute harmonic tones are remarkable for their peculiar ſoftneſs and ſweetneſs, whence they are ſometimes called flute tones, from the ſimilitude they bear to the notes of that inſtrument. This peculiarity in their quality may be probably accounted for in the following manner: When tones are produced on any ſtringed inſtrument by a bow, it acts on the ſtrings, by means of the adhering reſin, in the ſame manner as the teeth of a wheel. We may therefore conclude, that the vibrations of the ſtring muſt be interrupted and rendered conſiderably irregular by this friction; and the mo violent the friction, the greater muſt be the irregularity and diſturbance of the vibrations, and conſequently the harſher and more grating muſt be the tone: which we find by experiment to be the caſe. Now the harmonic tones, on this principle, ſhould be peculiarly ſoft and delicate, both becauſe they are generated by a very ſlight and gentle friction; and alſo, principally, becauſe, there is but one of the ſeveral aliquot portions of the ſtring which produce them, on which any friction at all is exerted. If this be the true ſolution, we ſee why wind inſtruments ſhould produce ſweeter and ſofter tones than ſtringed inſtruments, namely, becauſe their tones are generated without friction. And this is not only true in wind inſtruments, but in ſtringed inſtruments alſo, which produce tones as ſoft as thoſe of wind inſtruments, when theſe tones are generated without friction, as is the caſe in the harp of Æolus. In the muſical glaſſes indeed, and in the inſtrument called the Harmonica, the tones, though remarkable for their ſoftneſs, are produced by friction; but the ſurface of the finger is ſo ſmooth, that no conſiderable interruption can take place in the vibrations of the glaſs; and we may obſerve, that the tone is of a much finer quality, after the removal of the finger, while the vibrations of the glaſs continue; but if the ſurface of the finger be rendered rough, by moiſtening it with an acid, the tone will be rendered in a conſiderable degree harſh and grating.

SECT. V.

OF THE HARP OF ÆOLUS.

71. THE phænomena of the Æolian Lyre may be accounted for on principles analogous to thoſe by which we have attempted to explain the phænomena of ſympathetic tones in ſection 2. This pleaſing inſtrument, which has been reputed by ſome to be a modern diſcovery, was in truth the invention of Kircher, who has treated largely of it in his Phonurgia, (*ſee page* 148). It is an inſtrument ſo univerſally known, that it may well be preſumed unneceſſary to give any account either of its conſtruction, or the manner of uſing it.

To remove all uncertainty in the order of the notes in the lyre, I took off all the ſtrings but one; and on placing the inſtrument in a due poſition, was ſurpriſed to hear a great variety of notes, and frequently ſuch as were not produced by any aliquot part of the ſtring; often too I heard a chord of two or three notes from this ſingle ſtring. From obſerving theſe phænomena, they appeared to me ſo very complex and extraordinary, that I deſpaired of being able to account for them on the principle of aliquot parts. However, on a more minute enquiry, they all appeared to flow from it naturally and with eaſe.

But before we proceed to examine the phænomena, let us conſider what will be the effect of a current of air ruſhing againſt a ſtretched elaſtic fibre. The particles which ſtrike againſt the middle point of the ſtring will move the whole ſtring from its rectilineal poſition, and as no blaſt continues exactly of the ſame ſtrength for any conſiderable time, although it be able to remove the ſtring from its rectilineal poſition, yet, unleſs it be too rapid and violent, it will not be able to keep it bent; the fibre will therefore, by its elaſticity, return to its former poſition, and by its acquired velocity paſs it on the other ſide, and ſo continue to vibrate and excite pulſes in the air, which will produce the tone of the entire ſtring. But if the current of air be too ſtrong and rapid, when the ſtring is bent from the rectilineal poſition, it will not be able to recover it, but will continue bent and bellying like the cordage of a ſhip in a briſk gale. However though the whole ſtring cannot perform its vibrations, the ſubordinate aliquot parts may, which will be of different lengths in different caſes, according to the rapidity of the blaſt. Thus when the velocity of the current of air increaſes, ſo as to prevent the vibration of the whole ſtring, thoſe particles which ſtrike againſt the middle points of the halves of the ſtring, agitate thoſe halves as in the caſe of ſympathetic and ſecondary tones; and as theſe halves vibrate in half the time of the whole ſtring, though the blaſt may be too rapid to admit of the vibration of the whole, yet it can have no more effect in preventing the motion of the halves, than it would have on the whole ſtring were its tenſion quadruple; for the times of vibrations in ſtrings of different lengths, and agreeing in other circumſtances, are directly as the lengths; and in ſtrings differing in tenſion, and agreeing in other circumſtances, inverſely as the ſquare roots of the tenſions (*ſee Smith or Malcolm*): and therefore their vibrations may become ſtrong enough to excite ſuch pulſes as will affect the drum of the ear: and the like may be ſaid of other aliquot diviſions of the ſtring. In the ſame manner as ſtanding corn is bent by a blaſt of wind, and if the wind be ſufficiently rapid, it will have repeated its blaſt before the ſtem of corn can recover its perpendicular poſition, and therefore will keep it bent. But if it decays in rapidity or ſtrength, the ſtem of corn will have time to perform a vibration before it is again impelled; and thus it will appear to wave backwards and forwards by the impulſe of the wind. Thoſe particles which ſtrike againſt ſuch points of the ſtring as are not in the middle of aliquot parts, will interrupt and counteract each others vibrations, as in the caſe of ſympathetic and ſecondary tones, and therefore will not produce a ſenſible effect. That we may be more fully perſwaded of the truth of theſe principles, I ſhall here ſet down the order of the æolian notes as

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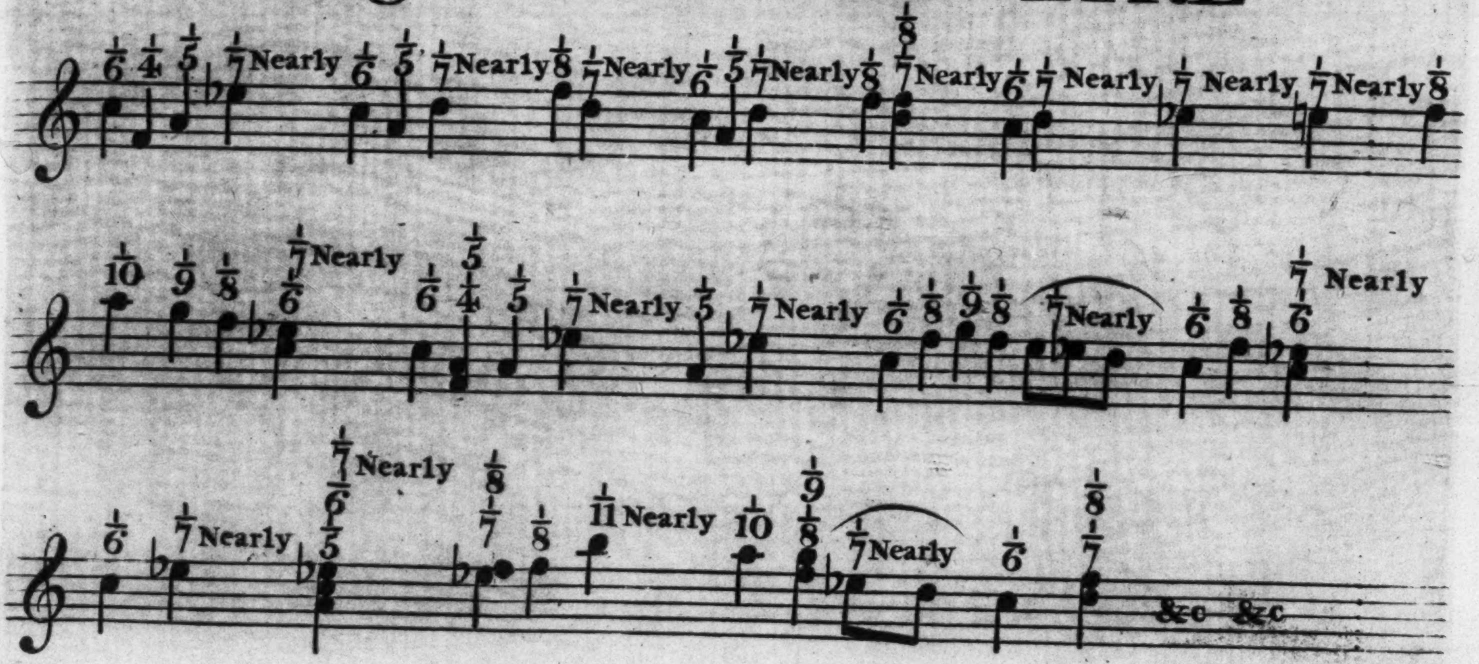
TABLE of Notes produced by a SINGLE ſtring of an ÆOLIAN LYRE


1/6 1/4 1/5 1/7 Nearly 1/6 1/5 1/7 Nearly 1/8 1/7 Nearly 1/6 1/5 1/7 Nearly 1/8 1/8 1/7 Nearly 1/6 1/7 Nearly 1/7 Nearly 1/7 Nearly 1/8
1/10 1/9 1/8 1/7 Nearly 1/6 1/6 1/5 1/4 1/5 1/7 Nearly 1/5 1/7 Nearly 1/6 1/8 1/9 1/8 1/7 Nearly 1/6 1/8 1/7 Nearly 1/6
1/6 1/7 Nearly 1/7 Nearly 1/6 1/5 1/8 1/7 1/8 1/11 Nearly 1/10 1/9 1/8 1/7 Nearly 1/6 1/8 1/7 &c &c

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accurately as a good ear could diſcover.

72. *Obſervation* 1. The original note of the ſtring being the grave fifteenth to low F on the violin, the æolian notes, as given in the annexed page, were diſtinctly perceived, and nearly in the ſame order in which they are ſet down.

From the table of proportions in Smith's Harmonics, p. 10, we may ſee, that theſe notes were produced by ſuch aliquot parts of the ſtring as are denoted by the fractional indexes which are written over them, agreeable to the theory laid down.

73. *Obſervation* 2. While ſome of theſe notes were ſounding, I applied an obſtacle indifferently to any point which divided the ſtring into ſuch aliquot parts as would produce theſe notes,

and the æolian note was not interrupted: but if I placed it on any other part, the tone was inſtantly extinguiſhed. This evidently ſhews, that the entire ſtring is in fact reſolved into ſuch parts, as from the preceding chain of reaſoning we ſhould have been induced to preſcribe for it.

74. *Obſervation* 3. I applied an obſtacle ſlightly againſt the ſtring, ſo as that its diſtance from the extremity ſhould be an aliquot part of the whole; and the æolian note was that which would be produced by ſuch an aliquot part: thus we may in general pre-determine what note the harp ſhall ſound. But this effect will not invariably take place; becauſe, though the obſtacle may determine the ſtring to reſolve itſelf into ſuch aliquot parts rather than any others, yet the blaſt may be too ſtrong or too weak, to admit of ſuch a part's vibrating with

ſufficient ſtrength to produce a ſound; however, if any note be produced in this caſe, it muſt either be that of this very aliquot part, or of ſome of its own aliquot diviſions; for the obſtacle muſt neceſſarily determine one of the interſections of the equal indentures.

75. *Obſervation* 4. When the blaſt riſes or falls, we find the tone alſo gradually riſe or fall: becauſe, as the blaſt riſes, it grows too ſtrong to admit of the vibrations of the longer aliquot parts; the vibrations of the ſhorter aliquot parts therefore will predominate, and will gradually ſhorten as the blaſt riſes in ſtrength. But in caſes of ſudden variations in the ſtrength of the blaſt, there will be alſo ſudden tranſitions in the tones.

76. *Obſervation* 5. We ſometimes hear a chord conſiſting of two or

three æolian notes; becauſe the blaſt which is of ſuch a degree of ſtrength as to admit of the vibrations of certain aliquot parts, may alſo admit of the vibrations of other parts, if they be not very different in length; for their vibrations will be performed in times not very different. But if the lengths of theſe parts, and conſequently their times of vibration, be very different, the blaſt that admits of the vibration of the one will prevent that of the other. Accordingly, in looking over the foregoing table, we find that the chords conſiſt of thoſe notes which are produced by ſuch different aliquot parts as are leaſt unequal: thus, one chord conſiſts of C and E, which notes are produced by one-ſixth and one-ſeventh of the ſtring. Another chord conſiſts of F and A, which are produced by one-fourth and one-fifth of the ſtring. Another conſiſts of A, C and E, which notes are produced by one-fifth, one-ſixth and one-ſeventh parts of the ſtring.

It is alſo worthy of obſervation, that in long ſtrings we never hear the original note and its octave at the ſame time; becauſe though they are the next aliquot parts, yet their difference is ſo great, that the blaſt which admits of the vibration of one of them, will obſtruct and prevent the other. It is only in the higher diviſions of the ſtring that the chords are heard at all; and the ſharper the note, the more frequent are the chords, for the reaſons aſſigned above, namely, becauſe the different aliquot parts, in ſuch caſes, approach nearer to equality.

77. *Obſervation* 6. Æolian tones are often heard, which are not produced by any exact ſubmultiple of the ſtring: but ſuch notes are very tranſi-

tory, and immediately vary their pitch, gradually falling or riſing to the notes next below or above them, which are produced by exact aliquot parts of the whole ſtring. This ariſes from the tranſition of the diviſions of the ſtring from one number to another; for during this tranſition, the parts of ſtring, whoſe vibrations produce the note, are gradually lengthening or ſhortening. Thus, ſuppoſe the æolian tone was produced by one-third of the ſtring AB, and that the breeze ſo varies as to cauſe this tone to fall into the octave of the original note; the points C and D (*fig.* 22), will gradually run towards *n*, and by ſo doing, will produce a note gradually flattening until it terminates in the octave to AB.

78. Diſcords are alſo often heard from the uniſon ſtrings of this inſtrument; the cauſe of this is alſo evident from the manner in which the notes are ge-

nerated. For the aliquot parts of a ſtring contain in themſelves an infinite variety of diſcords. *See page* 92.

79. Kircher, in his *Phonurgia*, page 148, has attempted to account for theſe phænomena of the æolian lyre, by ſuppoſing the current of air to ſtrike on different portions of the ſtring. But this is abſolutely overturned by experience; for ſuppoſe the æolian note to be one-fifth above the original note of the ſtring; that is, produced by AC, one-third of the whole AB (*fig.* 22); then, according to Kircher, the remaining part CB would be at reſt, which is falſe; for an obſtacle applied to any other point than C or D will deſtroy the æolian tone. Beſides, the chords that would ariſe on this theory, are not ſuch as really take place in nature: thus, where the chord conſiſts of the notes F and A (*ſee the plate for the æolian lyre*), the firſt note F is produced, according

to Kircher, by the blaſt's ſtriking on one-fourth of the ſtring; now in this caſe the remaining part of the ſtring muſt be at reſt, according to Kircher, but contrary to experience; or if it be agitated as one ſtring, it muſt produce the note of three-fourths of the whole ſtring; that is, a fourth above the baſe note; whereas the note really produced is the double octave to the third above the baſe note, as may be ſeen in the table of the æolian tones.

SECT. VI.

OF GRAVE HARMONIC TONES.

80. BESIDES the harmonic tones treated of in ſect. 4, there is another ſpecies of them which may be called *grave harmonics*, becauſe they are graver than the principal tones which generate them. Theſe tones were diſcovered by Sig^r. Tartini. However, M. D'Alembert informs us, that M. Romieu had made the ſame diſcovery a ſhort time before him. Tartini, who founded his ſyſtem of harmony on theſe grave harmonics, in the ſame manner as Rameau derived his from the acute, has given an account of his experiments, in his *Trattato di Muſica ſecundo la vera ſcienza*

dell' armonia, printed at Padua, 1754. The original work I have not ſeen, but have taken the following extract reſpecting it from the *Encyclopædie*, article *Fondamental*; and from *Rouſſeau's Muſical Dictionary*, article *Syſteme*.

81. "Two ſounds (ſays Tartini) being produced at the ſame inſtant, by an inſtrument capable of holding the note, that is, continuing it for ſome time without intermiſſion, as a trumpet, hautbois, violin, &c. theſe ſounds will produce a third ſound ſufficiently ſenſible. Thus, if on the ſame violin, and at the ſame inſtant, you produce two ſtrong and continued ſounds, in any relation whatſoever to each other, they will produce a third ſound, which will be different in different intervals of the principals. The ſame thing will happen if, inſtead of producing theſe two ſounds from the ſame violin, you produce them ſeparately,

but at the ſame inſtant, from two violins diſtant from each other about five or ſix paces. The hearer being ſituated between the inſtruments will perceive the third ſound ; and that the more diſtinctly, the nearer he is to the middle point between them. This experiment ſucceeds better with hautbois or wind-inſtruments, than with violins.

The third ſound produced in different caſes is as follows :

Two ſounds at the interval of a fifth, generate the uniſon of the grave generator.

A fourth minor generates the lower octave to the acute, or the fifth to the grave generator.

A third major, the lower octave to the grave.

A ſixth minor, the double octave to the acute.

A third minor, the tenth major of the grave.

A ſixth major, the tenth major of the acute.

A tone major, the double octave to the grave.

A tone minor, the double octave to the third major of the acute.

A ſemitone major, the triple octave of the acute.

A ſemitone minor, the twenty-ſixth of the grave."

He has alſo obſerved a ſingular conſequence which follows from theſe experiments, *viz.* that if the generating ſounds be

ut, *ſol*, *ut*, *mi*, *ſol*, *ut*, *re*, *mi*, *ſol*, *ſi*, *ut*,
$\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{9}$ $\frac{1}{10}$, $\frac{1}{12}$, $\frac{1}{15}$, $\frac{1}{16}$,
ſol, *ſol*, ⁕, &c.
$\frac{1}{24}$, $\frac{1}{25}$,

the third note reſulting from any two conſecutive notes in this progreſſion will be the ſame, *viz.* the loweſt $\underset{\frac{1}{2}}{ut}$ in the ſeries. He alſo obſerves, that the uniſon and octave are the only intervals which do not produce a third ſound.

82. Theſe are the phænomena of Sig^{r}. Tartini's *grave harmonics:* however the third ſound produced in all theſe different caſes ſhould be depreſſed an octave, as was determined at a meeting of ſeveral accurate judges, before whom theſe experiments were carefully repeated. And this correction is conformable to the account

which M. Romieu preſented to the Academy of Sciences at Montpelier, in which he informs us, that he found the third tone to be that of a ſounding body, whoſe number of vibrations was the greateſt common meaſure of the numbers denoting the ſynchronal vibrations of the generators; that is, an octave lower than according to Tartini. The phyſical ſolution of theſe phænomena ſhall be now propoſed.

83. The *acute harmonics* are derived from ſome ſubmultiple reſolution of the ſounding body, as we have ſeen in a former ſection. But the *grave harmonics* cannot by any means be accounted for by conſidering the ſounding body itſelf; as either a total or partial vibration thereof would produce either a uniſon, or an acuter note, and therefore cannot generate this ſpecies of harmonics, which are graver than the principal ſounds. We muſt therefore have re-

courſe to the aerial pulſes, as the only remaining cauſe which can contribute towards producing theſe effects.

84. Two notes of any interval being ſounded together, it is evident, that if one of the two extreme vibrations of each of the principals coincide, the remaining vibrations will not. Thus, for example, if the interval be a major third, the number of vibrations performed in the ſame time, by the acute and the grave, will be reſpectively five and four. Now if the firſt vibration of the grave coincides with the firſt of the acute, the fifth vibration of the grave will coincide with the ſixth of the acute; becauſe the one performs four vibrations while the other performs five. Thus if the equal portions of time, in which the generators perform their proper number of vibrations, be repreſented by the equal right lines AB, BC, CD, &c. and MN, NO, OR, &c. (*fig.* 30),

and theſe lines be ſubdivided into five and four equal parts reſpectively, the middle points *a*, *b*, *c*, *d*, *e*, of theſe ſubdiviſions, and *m*, *n*, *o*, *p* will denote the moments of time in which the pulſes are ſucceſſively produced by theſe vibrations. Now ſince *mr* is equal to MN; that is, to AB; that is, to *af*; if *m* be coincident with *a*, *r* will be coincident with *f*: and none of the intermediate pulſes will coincide; for if *p*, for inſtance, were coincident with *e*, then *mp*, which is equal to MP, would be equal to *ae*; that is, to AE; and therefore the number of ſynchronal vibrations, performed by the generators, would not be as four to five. That is, if the firſt pulſe of the grave generator be coincident with the firſt of the acute, the fifth pulſe of the grave will be coincident with the ſixth of the acute; but the intermediate pulſes will ſtrike ſeparately on the ear. Now at thoſe inſtants in which the ear is ſtruck by the coincident pulſes, it will be more affected than at any other. And therefore, beſides the two ſtrong ſenſations produced by the pulſes propagated by the vibrations of the ſounding bodies, there will be another additional ſenſation of one impulſe on the ear ſtronger than any others, while the principals reſpectively perform the number of vibrations denoted by the terms of their ratio. And therefore the effect will be preciſely the ſame as if the coincidences did not increaſe the intenſity of the extreme vibration, but that this effect was produced by any other means; as for inſtance, by the feeble vibration of a third body, which ſhould perform exactly one vibration while the principals performed their proper number; for every vibration of this body would coincide with one of the extreme vibrations of the principals, and conſequently increaſe their intenſity in the ſame manner as the coincidences. The third ſound therefore generated in all caſes, ſhould be that of a body which would vibrate once, while the principals performed their proper vibrations.

85. Let us examine now, whether the ſound, which would ariſe according to this theory, is ſuch as is in reality produced, conformably to M. Romieu's experiments and our own.

In the interval of a fifth, the number of vibrations performed in the ſame time by the acute and grave are three and two; the third ſound therefore in this caſe, and in all others alſo except the octave, being ſuch as would be produced by a body vibrating once in the ſame time, ſhould be an octave to the grave principal, one to two being the ratio of octaves: which is confirmed by experiment.

In a fourth minor, the vibrations of the principals are four and three. The third ſound therefore ſhould be ſuch as would be produced by one vibration in the ſame time that the principals perform four and three; therefore it ſhould be the double lower octave to the acute, one to four being the ratio of a double octave; or it ſhould be the twelfth to the grave, one to three being the ratio of a twelfth: which is alſo confirmed by experiment.

In a third major, the vibrations of the principals are five and four; the third ſound therefore ſhould be the double lower octave to the grave, one to four being the ratio of a double octave: which is confirmed by experiment.

In a ſixth minor, the principal vibrations are eight and five; the third ſound ſhould be therefore the triple octave to the acute principal, one to eight being compounded of the ratios one to two, two to four, and four to eight, that is, three octaves.

In a third minor, the principal vibrations are ſix and five. Now one to five is compounded of the ratios of one to two, two to four, and four to five; that is, of two octaves and a third major, that is, a ſeventeenth. Therefore the third ſound ſhould be a ſeventeenth major to the grave principal: which is conformable to experience.

In a ſixth major, the principal vibrations are five and three; but one to five is a ſeventeenth major as before. Therefore the third ſound ſhould be a ſeventeenth major to the acute principal: which is conformable to experience.

In a tone major, the principal vibrations are nine and eight; but one to eight is compounded of the ratios one to two, two to four, and four to eight, that is, three octaves. Therefore the third ſound ſhould be a triple octave to the grave principal: which is confirmed by experience.

In a tone minor, the principal vibrations are ten and nine; but one to ten is compounded of the ratios one to four, four to five, and five to ten, that is, of a double octave, a major third and an octave. Therefore the third ſound ſhould be the triple octave to the major third of the acute principal: which is conformable to experience.

In a major ſemitone, the principal vibrations are ſixteen and fifteen; but one to ſixteen is compounded of the ratios one to two, two to four, four to eight, and eight to ſixteen, that is, four octaves. Therefore the third ſound ſhould be the quadruple octave to the acute principal: which is conformable to experience.

In a minor ſemitone, the principal vibrations are twenty-five and twenty-four; but one to twenty-four is compounded of the ratios one to eight, that is, a triple octave; and of eight to twenty-four, or one to three; that is, an octave to the fifth, or a twelfth. Therefore the third ſound being compoſed of a triple octave and a twelfth, ſhould be a thirty-third to the grave principal: which is conformable to experience.

86. Tartini obſerved, as I mentioned above, that the intervals of the uniſon and octave produce no third ſounds; the reaſon of which is evident from the preceding theory. For in the uniſon, every ſucceſſive vibration of the one ſounding body coincides with every

ſucceſſive vibration of the other; conſequently all the ſucceſſive impulſes on the ear are of the ſame magnitude; and therefore there cannot be any third ſound generated. In the octave, every vibration of the grave coincides with every alternate vibration of the acute; and therefore every alternate impulſe on the ear will be ſtronger than that immediately preceding it. Now this effect would tend to generate the ſenſation of an octave below the acute, or the uniſon of the grave; which effect will therefore be inſenſible, it being very difficult to diſtinguiſh octaves or uniſons when ſounding together; but particularly in this caſe, where the tone is ſo exceedingly feeble.

The third ſound reſulting from the interval of the minor fourth, and from the two thirds, and two ſixths, whether major or minor, is the moſt eaſily diſtinguiſhed by the ear; becauſe there is always one of the principals to which it is not an octave, and it is not in uniſon with either of them. But the third ſound reſulting from a fifth, is perceived with difficulty, becauſe it is an octave to the grave principal. And it is diſtinguiſhed with more difficulty in the major and minor tones, becauſe as they differ but little from each other, the intonation eaſily confounds them; and the difficulty in the major and minor ſemitones is ſtill greater, their difference being ſtill leſs.

87. If three principal notes were ſounded together, equally ſtrong and even, a very delicate ear would diſtinguiſh three of theſe grave harmonics at the ſame time; which are produced according to the law aſſigned above, from the united effect of each of the two extreme principal ſounds with the intermediate, and with each other. Some of theſe grave harmonics would, in ſome caſes,

be the ſame; in others they would be different. Thus, if the ſynchronal vibrations performed by the three bodies were two, three, and ſix, the combination of the firſt and ſecond would produce the octave below the firſt; the combination of the ſecond and third would produce the uniſon of the ſecond; and the combination of the firſt and third would produce the ſame third ſound as the combination of the firſt and ſecond. And were the human ear ſtill more delicate, it would perceive diſtinct ſounds ariſing from the coincidences of the pulſes of theſe ſecondary ſounds with thoſe of the principals and of each other, and ſo on *ad infinitum.*

88. It has been very acutely objected to the doctrine of coincidences, that if the firſt pulſes propagated from the ſounding bodies do not arrive at the ear at the ſame inſtant, they will in ſome caſes never coincide. This we muſt ac-

knowledge is true; but then there will in ſuch caſes be always two pulſes which will very nearly approach to an actual coincidence, and as the effects of all impreſſions on the organs of ſenſe are not inſtantaneous, but continue for ſome time, theſe pulſes, when they make their neareſt approach to each other, will have the ſame effect as if they had actually coincided: and after they have made this their neareſt poſſible approach, they will continue afterwards to preſerve that interval, becauſe the two bodies perform their proper vibrations in the ſame time, and conſequently the ſame impreſſion will return as regularly as if they had actually coincided.

89. Tartini obſerved, as is mentioned above, that any two conſecutive ſounds in the progreſſion, *ut* ($\frac{1}{2}$), *ſol* ($\frac{1}{3}$), *ut* ($\frac{1}{4}$), &c. would always produce the ſame third ſound, *viz.* the octave to *ut* ($\frac{1}{2}$),

the loweſt in the ſeries. The reaſon of this is evident from the principles now laid down; for the number of ſimultaneous vibrations of theſe notes are denoted by the denominators of the fractions of this ſeries: ſuppoſe then the two principals ſounded together are *mi* ($\frac{1}{10}$) and *ſol* ($\frac{1}{12}$); while *ſol* ($\frac{1}{12}$) vibrates twelve times, *mi* ($\frac{1}{10}$) vibrates ten times; but the third ſound ariſing from *mi* and *ſol* is ſuch as would be produced by a body vibrating once, while *mi* vibrates ten times; and therefore will be the octave below *ut* ($\frac{1}{2}$); and the ſame argument holds in all other caſes.

90. M. D'Alembert invited the muſicians to determine what third ſound would ariſe from the interval of a triton, and a falſe fifth. M. Romieu anſwered, from experiment, that the third ſound reſulting from the latter interval, was the quintuple octave to the acuteſt of

the principals; contrary to his own law of the grave harmonics. For the ratio of a falſe fifth is ſixty-four to forty-five; and unity, as before, denotes the ſimultaneous vibration of the ſounding body which would produce the third ſound. Now the ratio of two to ſixty-four is compounded of the following ratios, one to two, two to four, four to eight, eight to ſixteen, ſixteen to thirty-two, and thirty-two to ſixty-four, that is, ſix octaves. But in ſuch delicate ſounds as theſe, it is a very difficult matter to diſtinguiſh a note and its octave, which are often confounded in caſes where they are much more ſenſible than in theſe grave harmonics. Romieu has not determined what third ſound reſults from a triton or redundant fourth: according to the preceding theory it muſt be a quintuple octave to the grave principal; for the ratio of this interval is forty-five to thirty-two; but the ratio of one to thirty-two is compounded of the following ratios, one to two, two to four, four to eight, eight to ſixteen, and ſixteen to thirty-two, that is, five octaves.

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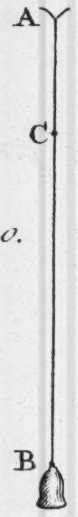

Fig. 20.

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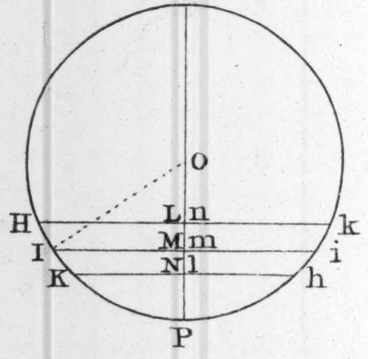

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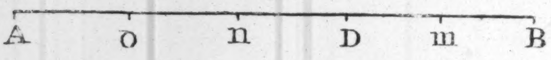

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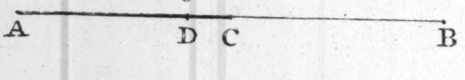

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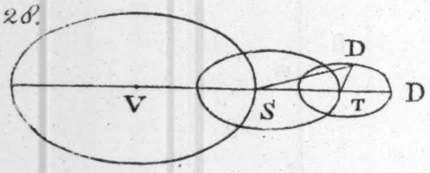

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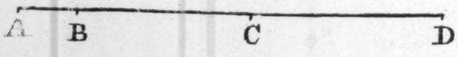

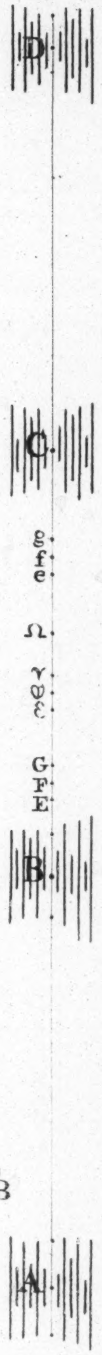

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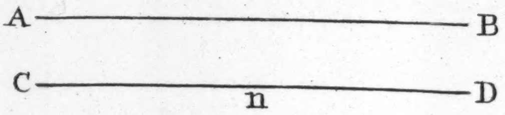

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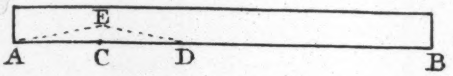

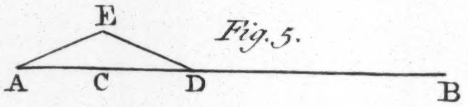

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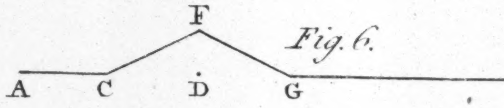

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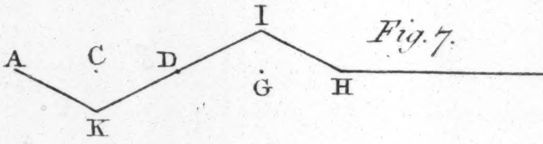

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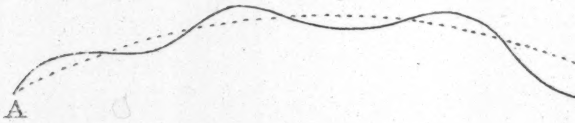

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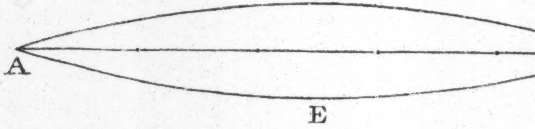

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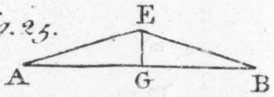

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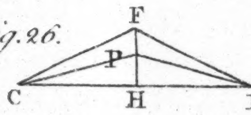

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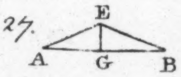

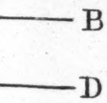

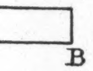

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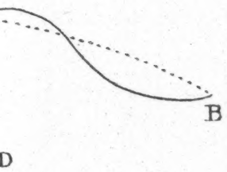

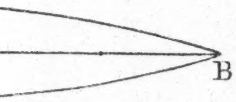

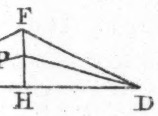

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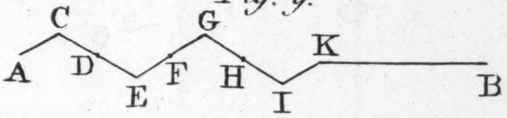

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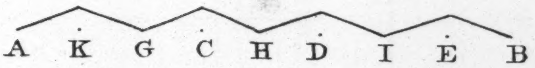

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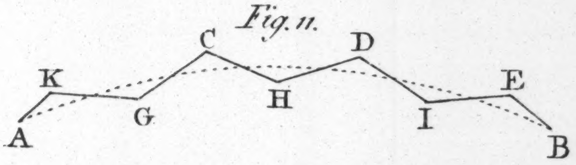

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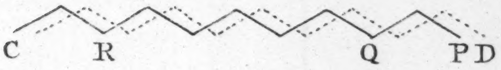

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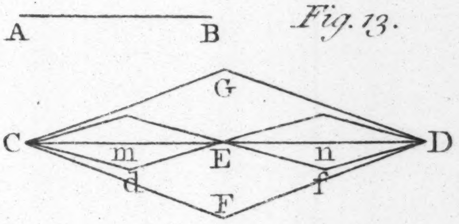

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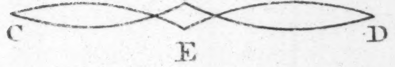

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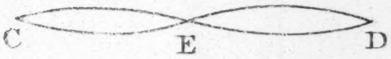

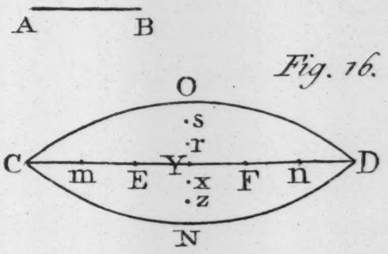

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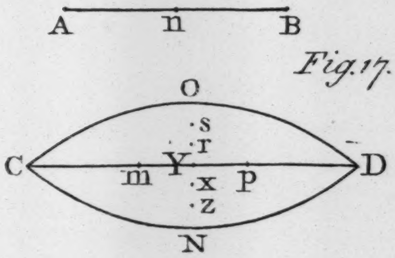

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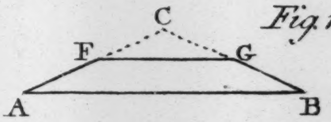

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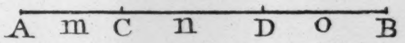

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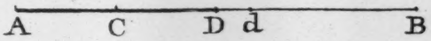

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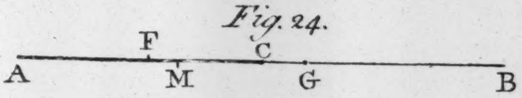

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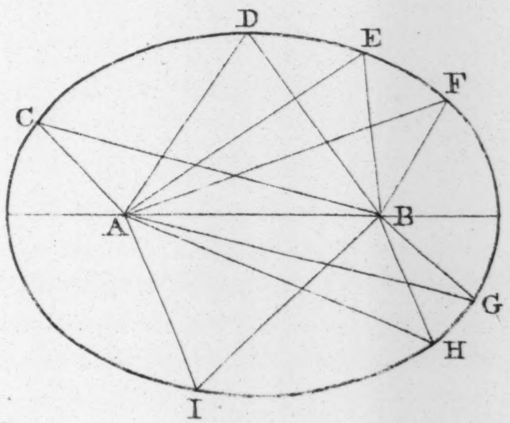

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